# LINEAR PROGRAMMING

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# PREFACE

The increased use of mathematics and statistics in solving business and industrial problems focuses attention on linear programming as one of the few sophisticated analytical devices finding wide acceptance in industry. A number of universities have introduced courses in linear programming, but, until recently, there was no suitable textbook. Most of them either avoided entanglement with the mathematical aspects of linear programming completely, or else discussed them at a level comprehensible only to the mathematically initiated. To a growing group of non-mathematically-oriented people, both on and off campus, who heed the trend and therefore wish to understand the theory of linear programming, the lack of an effective vehicle of learning remains a continued source of frustration.

This book serves as an introduction to the subtle mathematical reasoning that underlies and gives strength to linear programming as an analytical tool. Thus the reader should emerge with a somewhat enlarged stock of mathematical knowledge, as a result of having studied this book, would gratify, rather than surprise, the author.

Assuming only a knowledge of elementary algebra, the book begins by explaining the mathematical characteristics of a linear programming problem. This is followed by a careful explanation of the rudiments of matrix algebra and determinants, and such basic concepts as linear independence, vector space, and basis. In Chapter III the mathematical properties of linear programming solutions are discussed so as to pave the way for the development of the simplex method. The method itself is treated in Chapters IV and V which deal with its theoretical and computational aspects respectively. Two important variations of the simplex method, the revised simplex and the dual simplex, are introduced in Chapter VI. Sensitivity analysis is taken up in Chapter VII and is illustrated with practical examples. The theory of the dual is developed in Chapter VIII by employing the novel concept of a vector solution pair.

With much of the basic theory out of the way, the transportation problem is introduced as a special case of the general linear programming problem. Finally, in Chapter X, a number of the more advanced linear programming topics are described.

Numerical examples throughout the text illustrate important concepts and theorems. Examples with business or industrial content are used whenever possible so that the reader will constantly be reminded of the practical relevance of linear programming. No attempt is made to circumvent concepts and theorems just because they are mathematical. On the contrary, continuous effort is made to deal with them simply and clearly, but with increasingly liberal use of mathematics consistent with the expected development of the reader's mathematical ability. Mathematics is basically an essay in logic, and what is logical should not be difficult to understand once its baffling symbols and fundamental precepts are firmly grasped. As intimated earlier, one of the principal aims of this book is to sharpen the reader's mathematical facilities to a point where, upon completion of this volume, he may venture into technical journals in this field.

Students whose major interests lie in areas such as operations research, industrial engineering, management science, and applied economics, whether they be undergraduate or graduate students, will find this book especially designed for use as a text. Exercises of a varied nature, such as those that illuminate theory and those that develop skill, are included in each chapter, thus making this book suitable also for managerial and staff personnel in industry as well as for anyone else who is willing to use the self-study method. Tested in a quarterly course at Drexel for two years, it has been found to contain more material than can be handled comfortably in one quarter. If time is a limiting factor, then one may omit, without loss of continuity, the following material: Sections 6–5, 6–6 of Chapter VI, Chapter VII, Section 8–4 of Chapter VIII, and Chapter X. Students with adequate background knowledge of matrix algebra, however, should have no difficulty in covering the whole book simply by substituting Chapter II for the omissions suggested above. One advantage of this book is that it permits individual instructors to add to it in whatever way they deem desirable. For instance, those who favor theory may introduce additional materials on network flow theory, whereas others who favor applications may expose the students to more industrial examples found in current literatures.

Thanks are due to my colleagues at Drexel, particularly Samuel S. McNeary who read and offered important advices on the mathematical parts of the original manuscript. All errors are, of course, responsibilities of my own.

A. M. Chung

May, 1962

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# INTRODUCTION

## 1.1. General.

In the history of management technology, there is perhaps no other period more comparable in importance to the era of Frederick Taylor than the decade immediately following the Second World War. During this decade two important developments—of break-through proportions—paved the way for important advances in the essential art of management planning and control. One was the advent of the digital computer and the other was the application of higher mathematics and statistics to problems of industrial management. The former made possible rapid and economical manipulation of massive data, while the latter provided the necessary theoretical framework for the organization and analysis of these data. The result is that business problems of unprecedented complexity can now be solved and made a part of the rational decisions on which the success of businesses so importantly depends.

A vital part of these advances in management technology is the increasing application of linear programming to solve a wide range of managerial problems. This analytical tool has found ready adoption in many industries, including petroleum, chemicals, steel, and agriculture. Exploratory studies using the linear programming approach have also been made by airlines, railroads, utilities and financial institutions.

Moreover, interest in linear programming extends beyond the world of business. Economists, for instance, have found it helpful in attempting a re-appraisal of the theory of the firm and the theory of resource-allocation in a free-price economy. Mathematicians have through it discovered new avenues of research and investigation. In short, it is a subject that will be of considerable interest and importance to the business and academic worlds for some time to come. Our approach here, therefore, is to emphasize the fundamentals of the theory of linear programming, which, however, will be illustrated from time to time by problems relevant to the practical business world.

## 1.2. Characteristics of Linear Programming Problems.

Generally speaking, linear programming is a mathematical optimizing technique applicable to a class of problems having certain characteristics in common. The interpretation of these characteristics varies in accordance with the specific content of the problem. For this reason, it is best to describe these common characteristics in mathematical terms so as to retain their generality.

(1) A linear objective function—All linear programming problems have as their objective the optimization of some explicit linear function of many variables. If we denote this linear objective function by $f(X)$ and the appropriate variables of the problem by $x_1, x_2, \ldots, x_n$, then the goal of a linear programming problem is always to maximize (or minimize)

$$(1.2.1) \qquad f(X) = c_1x_1 + c_2x_2 + \ldots + c_nx_n{}^*,$$

where $c_1, \ldots, c_n$ represent the parameters of the problem. In other words, the goal is not just to accomplish something but to accomplish it in the "best" possible manner. The fact that business operations quite often are conducted with this type of goal in mind, e.g., to maximize profit, or to minimize cost, substantially explains why linear programming is found useful in business.

However, not all problems with optimizing objectives call for linear programming solutions. The objective must also be capable of being stated as a linear function of the variables of the problem. What do we mean by a linear function then? In mathematics, a function,

* Obviously the coefficients $c_1, \ldots, c_n$ are not restricted as to signs either individually or collectively.

say $f(X)$, is described as linear if and only if the following two conditions are satisfied:*

*a*. $f(kX) = kf(X)$,
*b*. $f(X_1 + X_2) = f(X_1) + f(X_2)$,

where $k$ is a constant coefficient; $X$, the set of variables, $x_1, x_2, \ldots, x_n$; and $X_1$, $X_2$, two different sets of values of $x_1, \ldots, x_n$. Stated in words, the first condition simply stipulates that multiplying the variables by a certain constant $k$ should result in the same functional value as multiplying the functional value of $X$ by the same constant $k$. The second condition is a little more difficult to explain. Essentially it requires that, given two sets of values of the variables, $X_1$ and $X_2$, the functional value of $(X_1 + X_2)$ must be the same as the sum of the two functional values obtained from $X_1$ and $X_2$ separatively.

Based on the above definition, we can easily verify that a first degree polynomial such as the one shown in (1.2.1) is a linear function. For if

$$f(X) = c_1x_1 + c_2x_2 + \ldots + c_nx_n,$$

then

$$\begin{aligned} f(kX) &= c_1(kx_1) + c_2(kx_2) + \ldots + c_n(kx_n) \\ &= k(c_1x_1 + c_2x_2 + \ldots + c_nx_n) \\ &= kf(X). \end{aligned}$$

Moreover, letting $X_1$ represent the set of values, $x_{11}, \ldots, x_{1n}$, and $X_2$ represent the other set of values, $x_{21}, \ldots, x_{2n}$, we have

$$\begin{aligned} f(X_1 + X_2) &= c_1(x_{11} + x_{21}) + c_2(x_{12} + x_{22}) + \ldots + c_n(x_{1n} + x_{2n}) \\ &= (c_1x_{11} + c_2x_{12} + \ldots + c_nx_{1n}) \\ &\qquad + (c_1x_{21} + c_2x_{22} + \ldots + c_nx_{2n}) \\ &= f(X_1) + f(X_2). \end{aligned}$$

On the other hand, such functions as $\sin x_1 + \sin x_2$, $x_1^2 + x_2^2$, or in general, all trigonometric functions, exponential functions and polynomials of higher degree than 1, are not linear functions because both conditions cannot be satisfied. Some functions may satisfy one but not the other condition. For instance, $f(X) = \dfrac{x_1^3 + x_2^3}{x_1^2 + x_2^2}$ satisfies the

* Throughout this text, the domain of real numbers is always assumed implicitly. In other words, functions and variables are always defined over the domain of real numbers, which is all that is necessary for the usual applications of linear programming.

first but not the second condition. Consequently, it is not a linear function.

(2) A set of linear constraints—Suppose we were to maximize a linear function such as $2x_1 - 3x_2$. Since $x_1$ is associated with a positive coefficient and $x_2$ with a negative coefficient, obviously we could increase the value of this function as long as we increase $x_1$ and decrease $x_2$. This could be continued indefinitely until $x_1$ is increased to positive infinity and $x_2$ decreased to negative infinity with the result that the value of the function would be positive infinity. As a matter of fact, all linear functions by themselves have a maximal value of positive infinity and a minimal value of negative infinity if no restrictions are placed on the values of their variables. This means that they all have the same maximum and minimum. Consequently, the problem of optimizing a linear function is not a mathematically meaningful problem unless the variables are constrained as to the ranges of values they may assume. In linear programming such restrictions are contained in a set of linear inequalities as follows:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2, \\
\vdots \qquad \vdots \qquad\qquad \vdots \quad & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m,
\end{aligned}
\tag{1.2.2}
$$

where $a_{11}, a_{12}, \ldots, a_{1n}, \ldots, a_{m1}, a_{m2}, \ldots, a_{mn}$ are constant coefficients* and $b_1, b_2, \ldots, b_m$ simply constants. There are three points that should be borne in mind in connection with these constraints (called *structural* constraints in linear programming terminology). First, they are linear in the same sense as the linear function is defined above. Second, they are characteristically inequalities although equality-constraints are not categorically excluded. Finally, the number of such constraints, i.e., $m$, is not restricted in any way except as it affects the practical problem of computation. Together these constraints define a region of acceptable values of the variables. Since this region may or may not include values up to infinity, linear programming becomes a problem of selecting a set of values (of the

* Since there are $m$ inequalities and $n$ variables, it is necessary to differentiate among the coefficients by double subscripts with the first referring to the inequality and the second to the variable. Thus, $a_{1n}$ is the coefficient of the $n$th variable ($x_n$) in the first inequality.

variables) within this region, which will yield the maximal (or minimal) value of the objective function.

The relevance of these mathematical expressions to the real world is not far to see. For example, if we are describing a management problem of deciding on how much of each of a group of $n$ products to produce, it is reasonable to expect that some of the products may require the same type of steel, or the same kind of welding operation, or the same assembling facilities. To the extent that the supply of raw material, or of machine time, or of assembling facilities is fixed, which is again reasonable to expect, there is an effective constraint on the relative production levels of these products. In addition, there may be time constraints imposed by delivery schedules, financial constraints imposed by budgets, technical constraints imposed by engineering considerations, and so on. As the next section will show, the question is not so much whether there are constraints or not, but whether they lend themselves to unambiguous mathematical statements and, if they do, whether the resulting statements are linear inequalities. If they are not, then the problem is not a linear programming problem.*

(3) The non-negativity constraints—As a direct outgrowth of its application in business and industry, all linear programming problems require that the solution values of their variables be non-negative, i.e., either zero or positive but not negative. This is necessary because the variables in business and industry, e.g., level of production, amount in inventory, etc. usually have no meaningful negative counterparts. Mathematically these non-negativity constraints are written as follows:

$$x_1, x_2, \ldots, x_n \geq 0. \tag{1.2.3}$$

Although at first glance the inclusion of these constraints seems to compound the difficulty of solving the problem, actually—as we will see in Chapter IV—they can be taken care of by a simple rule of transformation which automatically excludes all unacceptable negative solutions. Their presence in the problem has no more operational significance than the secondary advantage of providing an additional

* Current efforts to extend the simplex method of solving linear programming problems to programming problems with non-linear constraints are evidenced by Peter Wegner's article, "A Non-linear Extension of the Simplex Method," *Management Science*, VII (1960), pp. 43–56.

check on the accuracy of computations—a point which will become clear as we complete Chapter V.

In summary then, a linear programming problem is recognized by the following three parts:

$$\text{Maximize } f(X) = c_1x_1 + c_2x_2 + \ldots + c_nx_n, \qquad (1.2.1)$$

$$(1.2.4) \quad \text{subject to:} \quad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &\leq b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &\leq b_2, \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &\leq b_m, \end{aligned} \qquad (1.2.2)$$

$$\text{and} \qquad x_1, x_2, \ldots, x_n \geq 0. \qquad (1.2.3)$$

It may be well to point out that a "$\leq$" inequality may be converted into a "$\geq$" one if multiplied on both sides by $-1$. Hence, the expression of the structural constraints in (1.2.4) is perfectly general. Furthermore, since all inequalities can be converted into equalities by means of adding or subtracting non-negative variables, some prefer to write the constraints in the form of equations. In that event, however, it must be stipulated that $n \geq m$, or the problem may very well have no solution.

A casual survey of (1.2.4) would probably suggest that the problem could be solved by first finding all possible solutions to (1.2.2); then eliminating those which violate (1.2.3), and finally, selecting among the remaining solutions the one which optimizes $f(X)$ in (1.2.1). We soon realize, however, that (1.2.2) admits of an infinite number of solutions if any solution exists at all; therefore a complete enumeration of all of them is quite impracticable if not impossible. Accordingly, a more efficient method based on careful analysis of the theoretical properties of the problem is necessary. Such a method is the simplex method, first proposed by George B. Dantzig in 1947 in collaboration with Marshall Wood, Alex Orden and others while working on research projects in the U.S. Department of Air Force. This method together with its subsequent modifications, extensions and applications will constitute the subject matter of the ensuing chapters of this text.