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The Theory of Stochastic Processes III

Translated from the Russian
by S. Kotz

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Preface

It was originally planned that the *Theory of Stochastic Processes* would consist of two volumes: the first to be devoted to general problems and the second to specific classes of random processes. It became apparent, however, that the amount of material related to specific problems of the theory could not possibly be included in one volume. This is how the present third volume came into being.

This volume contains the theory of martingales, stochastic integrals, stochastic differential equations, diffusion, and continuous Markov processes.

The theory of stochastic processes is an actively developing branch of mathematics, and it would be an unreasonable and impossible task to attempt to encompass it in a single treatise (even a multivolume one). Therefore, the authors, guided by their own considerations concerning the relative importance of various results, naturally had to be selective in their choice of material. The authors are fully aware that such a selective process is not perfect. Even a number of topics that are, in the authors' opinion, of great importance could not be included, for example, limit theorems for particular classes of random processes, the theory of random fields, conditional Markov processes, and information and statistics of random processes.

With the publication of this last volume, we recall with gratitude our associates who assisted us in this endeavor, and express our sincere thanks to G. N. Sytaya, L. V. Lobanova, P. V. Boĭko, N. F. Ryabova, N. A. Skorohod, V. V. Skorohod, N. I. Portenko, and L. I. Gab.

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Chapter I

Martingales and Stochastic Integrals

§1. Martingales and Their Generalizations

Survey of preceding results. We start by recalling and making more precise the definitions and previously obtained results pertaining to martingales and semimartingales (cf. Volume I, Chapter II, Section 2 and Chapter III, Section 4).

Let $\{\Omega, \mathfrak{S}, \mathsf{P}\}$ be a probability space, let T be an arbitrary ordered set (in what follows only those cases where T is a subset of the extended real line $[-\infty, +\infty]$ will be discussed) and let $\{\mathfrak{F}_t, t \in T\}$ be a current of σ-algebras ($\mathfrak{F}_t \subset \mathfrak{S}$): if $t_1 < t_2$ then $\mathfrak{F}_{t_1} \subset \mathfrak{F}_{t_2}$. The symbol $\{\xi(t), \mathfrak{F}_t, t \in T\}$ or simply $\{\xi(t), \mathfrak{F}_t\}$ denotes an object consisting of a current of σ-algebras $\{\mathfrak{F}_t, t \in T\}$ on the measurable space $\{\Omega, \mathfrak{S}\}$ and a random process $\xi(t)$, $t \in T$, adopted to $\{\mathfrak{F}_t, t \in T\}$ (i.e., $\xi(t)$ if $\mathfrak{F}_t$-measurable for each $t \in T$). This object will also be referred to in what follows as a *random process*.

A random process $\{\xi(t), \mathfrak{F}_t, t \in T\}$ is called an $\mathfrak{F}_t$*-martingale* (or *martingale* if there is no ambiguity concerning the current of σ-algebras $\mathfrak{F}_t$ under consideration) provided

$$\mathsf{E}|\xi(t)| < \infty \qquad \forall t \in T \tag{1}$$

and

$$\mathsf{E}\{\xi(t) \mid \mathfrak{F}_s\} = \xi(s) \quad \text{for } s < t,\ s, t \in T,$$

it is called a *supermartingale* (*submartingale*) if it satisfies condition (1) and moreover

$$\begin{aligned} &\mathsf{E}\{\xi(t) \mid \mathfrak{F}_s\} \leq \xi(s), \qquad s < t, \quad s, t \in T \\ &(\mathsf{E}(\xi(t) \mid \mathfrak{F}_s\} \geq \xi(s), \qquad s < t) \end{aligned} \tag{2}$$

Observe that the above definition differs from that presented in Volume I since we now require finiteness of the mathematical expectation of the quantity $\xi(t)$ in all cases. Previously, in the case of the supermartingale, for example, only the finiteness of the expectation $\mathsf{E}\xi^-(t)$ was assumed.

The definition presented herein is equivalent to the following $\{\xi(t), \mathfrak{F}_t, t \in T\}$ is a martingale (supermartingale) if for any set $B_s \in \mathfrak{F}_s$ and for any s and t belonging to T such that $s < t$,

$$\int_{B_s} \xi(t) d\mathsf{P} = \int_{B_s} \xi(s)\, d\mathsf{P} \qquad \left(\int_{B_s} \xi(t)\, d\mathsf{P} \leq \int_{B_s} \xi(s)\, d\mathsf{P}\right)$$

Supermartingales and submartingales are also called *semimartingales*.

In this section we shall consider mainly semimartingales of a continuous argument.

The space of all real-valued functions on the interval $[0, T]$ which possess the left-hand limit for each $t \in (0, T]$ and which are continuous from the right on $[0, T)$ will be denoted by $\mathscr{D}$ or by $\mathscr{D}[0, T]$.

Analogous meaning is attached to the notation $\mathscr{D}[0, T)$, $\mathscr{D}[0, \infty)$, and $\mathscr{D}[0, \infty]$

A number of inequalities and theorems concerning the existence of limits plays an important role in the martingale theory The following relationships were established in Volume I, Chapter II, Section 2·

If $\xi(t)$, $t \in T$, is a separable submartingale, then

$$\mathsf{P}\left\{\sup_{t \in T} \xi^+(t) \geq C\right\} \leq \frac{\sup_{t \in T} \mathsf{E}\xi^+(t)}{C},$$

(4) $$\mathsf{E}\left[\sup_{t \in T} \xi^+(t)\right]^p \leq q^p \sup_{t \in T} \mathsf{E}[\xi^+(t)]^p, \qquad q = \frac{p}{p-1}, \quad p > 1,$$

(5) $$\mathsf{E}\nu[a, b) \leq \sup_{t \in T} \frac{\mathsf{E}(\xi(t) - b)^+}{b - a},$$

here $a^+ = a$ for $a \geq 0$ and $a^+ = 0$ for $a < 0$ and $\nu[a, b)$ denotes the number of crossings downward of the half-interval $[a, b)$ by the sample function of the process $\xi(t)$ (a more precise definition is given in Volume I, Chapter II, Section 2)

We now recall the definition of a closure of a semimartingale.

Let $\{\xi(t), \mathfrak{F}_t, t \in T\}$ be a semimartingale and let the set T possess no largest (smallest) element. The random variable η is called a *closure from the right (left) of the semimartingale* $\xi(t)$ if one can extend the set T by adding one new element b (a) which satisfies

$$t < b \ (t > a) \qquad \forall t \in T$$

and complete the current of σ-algebras $\{\mathfrak{F}_t, t \in T\}$ by adding the corresponding σ-algebra $\mathfrak{F}_b$ $(\mathfrak{F}_a)$ so th. the extended family of the random variables $\xi(t)$, $t \in T'$, $T' = T \cup \{b\}$ $(T' = T \cup \{a\})$ also forms an $\mathfrak{F}_t$-semimartingale.

Theorem 1. *Let $\xi(t)$, $t \in T$, be a separable submartingale, $T \subset (a, b)$, and the points a and b be the limit points for the set T $(-\infty \leq a < b \leq \infty)$. Then a set Λ of probability 0 exists such that for $\omega \notin \Lambda$:*

a) *in every interior point t of the set T the limits* $\xi(t-)$ *and* $\xi(t+)$ *exist*,

b) *if* $\sup\{\mathsf{E}\xi^+(t), t \in T\} < \infty$, *then the limit* $\xi(b-)$ *exists; moreover, if for some* t_0 *the family of random variables* $\{\xi(t), t \in [t_0, b)\}$ *is uniformly integrable, then the limit* $\xi(b-)$ *exists in* L_1 *as well and* $\xi(b-)$ *is a closure from the right of the submartingale*;

c) *if* $\lim_{t\to a} \mathsf{E}\xi(t) > -\infty$ *then the family of random variables* $\{\xi(t), t \in (a, t_0]\}$ *is uniformly integrable, the limit* $\xi(a+)$ *exists for every* $\omega \notin \Lambda$ *also in the sense of convergence in* L_1, *and* $\xi(a+)$ *is a closure from the left of the submartingale*

Proof. The existence with probability 1 of the one-sided limits $\xi(t-)$ and $\xi(t+)$ for each $t \in [a, b]$ under the condition that $\sup\{\mathsf{E}\xi^+(t), t \in (a, b)\} < \infty$ was established in Volume I, Chapter III, Section 4

Furthermore, if the family $\{\xi(t), t \in [t_0, b)\}$ is uniformly integrable then the convergence of $\xi(t)$ to $\xi(b-)$ with probability 1 as $t \uparrow b$ implies that this convergence is also valid in L_1 and that $\xi(b-)$ is a closure from the right of the submartingale $\xi(t)$, $t \in (a, b)$. This is proved in the same manner as in the case of submartingales of a discrete argument (Volume I, Chapter II, Section 2).

We need now to verify assertion c).

Let $l = \lim_{t\downarrow a} \mathsf{E}\xi(t)$. Since $\mathsf{E}\xi(t)$ is a monotonically nondecreasing function the existence of the limit is assured Moreover, the equality

$$|\xi(t)| = 2\xi^+(t) - \xi(t)$$

implies that

$$\sup_{t\in(a\ t_0]} \mathsf{E}|\xi(t)| \le 2\mathsf{E}\xi^+(t_0) - l = C < \infty$$

In view of Chebyshev's inequality $\mathsf{P}(B_t) \le C/N$, where $B_t = \{|\xi(t)| > N\}$, i e, $\mathsf{P}(B_t) \to 0$ as $N \to \infty$ uniformly in t Let $\varepsilon > 0$ be arbitrary and t_1 be such that $\mathsf{E}\xi(t) - l < \varepsilon/2$ for all $t < t_1$ Then for $t \in (a, t_1]$

$$\begin{aligned}\int_{B_t} |\xi(t)|\, d\mathsf{P} &= \int_{\{\xi(t)>N\}} \xi(t)\, d\mathsf{P} + \int_{\{\xi(t)>-N\}} \xi(t)\, d\mathsf{P} - \mathsf{E}\xi(t)\\ &\le \int_{\{\xi(t)>N\}} \xi(t_1)\, d\mathsf{P} + \int_{\{\xi(t)>-N\}} \xi(t_1)\, d\mathsf{P} - \mathsf{E}\xi(t)\\ &\le \int_{B_t} |\xi(t_1)|\, d\mathsf{P} + \frac{\varepsilon}{2},\end{aligned}$$

so that $\int_{B_t} |\xi(t)|\, d\mathsf{P} < \varepsilon$ for all $t \in (a, t_1]$ and N sufficiently large. Thus the family $\{\xi(t), t \in (a, t_0]\}$ is uniformly integrable The limit $\lim_{t\downarrow a} \xi(t)$ exists with probability 1, therefore it also exists in the sense of convergence in L_1

Now $\xi(a+)$ is indeed a closure from the left of the submartingale $\{\xi(t), t \in T\}$ This follows from the fact that it is permissible to approach the limit as $s \downarrow a$ under the integral sign in the inequality

$$\int_B \xi(s)\, d\mathsf{P} \le \int_B \xi(t)\, d\mathsf{P}, \qquad s < t, \quad B \in \bigcap_{t\in T} \mathfrak{F}_t \quad \square$$

Remark. It is evident that assertion c) of the theorem is valid for sequences also. In this case the assertion can be stated as follows.

If $\{\cdots\xi(-n),\xi(-n+1),\ldots,\xi(0)\}$ *is a submartingale and* $\lim_n \mathsf{E}\xi(-n)>-\infty$, *then the sequence* $\{\xi(-n)\}$ *is uniformly integrable, the limit* $\xi_\infty=\lim\xi(-n)$ *exists with probability* 1 *in* L_1 *as well and is a closure from the left of the submartingale* $\{\xi(n), n=\cdots -k,-k+1,\ldots,0\}$.

In what follows we shall call a semimartingale *uniformly integrable* if a corresponding family of random variables $\xi(t)$, $t\in T$, is uniformly integrable. We shall refer to a martingale as *integrable* if $\sup\{\mathsf{E}|\xi(t)|, t\in T\}<\infty$.

Theorem 2. *Let* $T\subset(a,b)$ *and let a and b be limit points of the set* T $(-\infty\leq a<b\leq\infty)$. *In order that martingale* $\{\xi(t),\mathfrak{F}_t, t\in T\}$ *be uniformly integrable, it is necessary and sufficient that a random variable* η *exist such that*

$$\mathsf{E}|\eta|<\infty,\qquad \xi(t)=\mathsf{E}\{\eta\,|\,\mathfrak{F}_t\},\qquad t\in T. \tag{6}$$

If this condition is satisfied, we can set $\eta=\lim_{t\uparrow b}\xi(t)$ *and the variable* η *will be uniquely determined* (mod P) *in the class of all* $\sigma\{\mathfrak{F}_t, t\in T\}$*-measurable random variables.*

Proof. By Theorem 1, if the martingale $\{\xi(t),\mathfrak{F}_t, t\in T\}$ is uniformly integrable, then it possesses a closure from the right and therefore admits representation (6).

Now let martingale $\xi(t)$ admit the representation given by formula (6). Then

$$\int_A \xi(t)\,d\mathsf{P}=\int_A\eta\,d\mathsf{P}\quad \forall A\in\mathfrak{F}_t,$$

which implies that

$$\int_B|\xi(t)|\,d\mathsf{P}\leq\int_B|\eta|\,d\mathsf{P}\qquad\forall B\in\mathfrak{F}_t. \tag{7}$$

In particular, $\mathsf{E}|\xi(t)|\leq\mathsf{E}|\eta|$. Therefore Chebyshev's inequality implies that $\mathsf{P}\{|\xi(t)|>N\}\to0$ as $N\to\infty$ uniformly in t. Applying inequality (7) to the set $B=B_t=\{|\xi(t)|>N\}$ we verify that the family $\{\xi(t), t\in T\}$ is uniformly integrable.

We need now to prove the uniqueness of representation (6) in the class of all $\sigma\{\mathfrak{F}_t, t\in T\}$-measurable random variables. Suppose that two such representations exist in terms of the random variables η_i, $i=1,2$. Then

$$\mathsf{E}\{\zeta\,|\,\mathfrak{F}_t\}=0\qquad\forall t\in T,$$

where $\zeta=\eta_1-\eta_2$.

Thus

$$\int_A\zeta\,d\mathsf{P}=0$$

for all A belonging to $\mathfrak{F}_t$ and all $t\in T$, and consequently for all A belonging to

$\sigma\{\mathfrak{F}_t, t \in T\}$. Since the variable ζ is $\sigma\{\mathfrak{F}_t, t \in T\}$-measurable it follows that $\zeta = 0$ (mod P). □

Remark. If $\{\xi(t), t \in T\}$ is a martingale and T contains a maximal element then the family of random variables $\xi(t)$, $t \in T$ is uniformly integrable.

A $\sigma\{\mathfrak{F}_t, t \in T\}$-measurable random variable η appearing in representation (6) is called *the boundary value of the martingale* $\xi(t)$, $t \in T$.

It was shown in Volume I, Chapter III, Section 4 that under very general assumptions there exists—for a given semimartingale—a stochastically equivalent process $\{\zeta(t), \mathfrak{F}_t, t \geq 0\}$ with sample functions belonging to $\mathscr{D}[0, \infty)$, and, moreover, the current of σ-algebras $\mathfrak{F}_t$ is continuous from the right, i.e.,

$$\mathfrak{F}_{t+} = \mathfrak{F}_t \qquad \forall t > 0.$$

In this section we shall assume, unless stated otherwise, that the semimartingales under consideration possess these properties.

Quasi-martingales. Let $\{\mathfrak{F}_t, t \geq 0\}$ be a current of σ-algebras continuous from the right ($\mathfrak{F}_t = \mathfrak{F}_{t+}$).

Definition. A process $\{\xi(t), t \geq 0\}$ adopted to $\mathfrak{F}_t$ is called a *quasi-martingale* (*$\mathfrak{F}_t$-quasi-martingale*) if

$$\mathsf{E}|\xi(t)| < \infty \qquad \forall t \geq 0$$

and

$$\sup \sum_{k=0}^{n-1} \mathsf{E}|\xi(t_k) - \mathsf{E}\{\xi(t_{k+1}) | \mathfrak{F}_{t_k}\}| = V < \infty,$$

where the supremum is taken over arbitrary values of n and $t_1, t_2, \ldots, t_n$, $0 \leq t_0 < t_1 < t_2 < \cdots < t_n < \infty$.

It will be shown below that a study of quasi-martingales can be reduced to a study of semimartingales.

As examples of quasi-martingales, one may note martingales, super-(sub)martingales for which $\inf \mathsf{E}\xi(t) > -\infty$ ($\sup \mathsf{E}\xi(t) < \infty$) and also processes which are differences of two supermartingales. It turns out that these examples exhaust all the possible quasi-martingales.

Set

$$\delta(s, t) = \xi(s) - \mathsf{E}\{\xi(t) | \mathfrak{F}_s\} \qquad (s < t), \quad a(t) = \mathsf{E}\xi(t).$$

Then

$$\sum_{k=0}^{n-1} |a(t_k) - a(t_{k+1})| = \sum_{k=0}^{n-1} |\mathsf{E}\delta(t_k, t_{k+1})| \leq V,$$

i.e., $a(t)$ is a function of bounded variation. In particular, for any $t>0$ there exist limits $a(t-)$ and $a(t+)$ and, moreover, $a(\infty)=\lim_{t\to\infty} a(t)$.

Inequalities (3)–(5) can be generalized for the case of quasi-martingales. For this purpose observe that inequalities (21) and (23) derived in Volume I, Chapter II, Section 2 for countable sequences can easily be adapted for separable quasi-martingales and can be written in this case in the form

$$\mathsf{P}\{\sup \xi(t) \geq C\} \leq \frac{\sup E\xi^{+}(t)+V}{C}, \tag{8}$$

$$\mathsf{E}\nu[a,b) \leq \frac{\sup_t \mathsf{E}(\xi(t)-b)^{+}+V}{b-a}, \tag{9}$$

where $\nu[a,b)$ is the number of downward crossings of the interval $[a,b)$. Utilizing inequality (9) one can prove the following theorem (analogously to the proofs of Theorems 6 and 7 for semimartingales in Volume I, Chapter III, Section 4).

Theorem 3. *A separable quasi-martingale $\xi(t)$, $t>0$, possesses with probability 1 for each t the left-hand and right-hand limits. Moreover, $\{\xi(t+), \mathfrak{F}_{t+}, t\geq 0\}$ is also a quasi-martingale with sample functions which are continuous from the right and $\mathsf{P}\{\xi(t)=\xi(t+)\}=1$ at each point t such that $\mathfrak{F}_t=\mathfrak{F}_{t+}$ and $\mathsf{E}\xi(t)$ is continuous.*

In view of this theorem, we can, without loss of generality, concentrate, in what follows, only on those quasi-martingales with sample functions belonging to $\mathcal{D}$ with probability 1 and for which $\mathfrak{F}_t=\mathfrak{F}_{t+}$ for all $t\geq 0$. In this subsection we shall assume that the stipulated conditions are satisfied.

Theorem 4. *An arbitrary quasi-martingale admits representation*

$$\xi(t)=\mu(t)+\zeta(t),$$

where $\mu(t)$ is a martingale and $\mathsf{E}|\zeta(t)|\to 0$ as $t\to\infty$.

This decomposition is unique.

If $\xi(t)$ is a supermartingale satisfying condition $\inf \mathsf{E}\xi(t)>-\infty$, then $\zeta(t)$ is a nonnegative supermartingale.

Proof. For each $s>0$ and $t\geq 0$ we set

$$\xi(s,t)=\mathsf{E}\{\xi(s+t)\,|\,\mathfrak{F}_t\},$$

and consider a separable modification of the process $\xi(s,t)$. We show that for a fixed t, $\xi(s,t)$ as a function of s is of bounded variation with probability 1.

Indeed

$$\sum_{k=0}^{n-1} |\xi(s_k,t)-\xi(s_{k+1},t)| = \sum_{k=0}^{n-1} |\mathsf{E}\{\xi(s_k+t)-\mathsf{E}\{\xi(s_{k+1}+t)|\mathfrak{F}_{s_k+t}\}|\mathfrak{F}_t\}|$$
$$= \sum_{k=0}^{n-1} \mathsf{E}\{|\delta(s_k+t, s_{k+1}+t)|\,|\,\mathfrak{F}_t\}$$

and

$$\mathsf{E}\sum_{k=0}^{n-1}|\xi(s_k, t)-\xi(s_{k+1}, t)|\leq V$$

It may be assumed that the set of separability points of function $\xi(s, t)$ in variables s and t is of the form $I\times I$. For each $t\in I$ we choose sequences $\{s_0, s_1, \ldots, s_n\}$, $s_k\in I$, such that these sequences—viewed as sets—increase monotonically as n increases and in the limit exhaust the whole set I. Moreover, the sums $\sum_{k=0}^{n-1}|\xi(s_k, t)-\xi(s_{k+1}, t)|$ are monotonically nondecreasing and approach their upper bound $V(t)$. Thus $\mathsf{E}\,V(t)\leq V$ and $V(t)<\infty$ with probability 1 for each $t\in I$. This implies the existence of a set $N\in\mathfrak{S}$ with $\mathsf{P}(N)=0$, such that if $\omega\notin N$ then $V(t)<\infty$ for *any* t. Thus, there exists with probability 1 the limit

$$\mu(t)=\lim_{s\uparrow\infty}\xi(s, t)$$

Let $s_n\uparrow\infty$. Since

$$|\mu(t)-\xi(s_n, t)|\leq\sum_{n}^{\infty}|\xi(s_k, t)-\xi(s_{k+1}, t)|\leq V(t),$$

$\mu(t)$ is an integrable random variable and the sequence $\xi(s_n, t)$ possesses an integrable majorant. Hence we have for $t_1<t_2$

$$\mathsf{E}\{\mu(t_2)|\mathfrak{F}_{t_1}\}=\mathsf{E}\left\{\lim_{s\to\infty}\mathsf{E}\{\xi(s+t_2)|\mathfrak{F}_{t_2}\}\mathfrak{F}_{t_1}\right\}$$

$$=\lim_{s\to\infty}\mathsf{E}\{\xi(s+t_2)|\mathfrak{F}_{t_1}\}=\mu(t_1)$$

Thus $\mu(t)$ is an $\mathfrak{F}_t$-martingale.

Now set $\zeta(t)=\xi(t)-\mu(t)$. Then $\mathsf{E}|\zeta(t)|\to 0$ as $t\to\infty$.

Indeed, assume the contrary. Then an $\varepsilon>0$ exists and for each $N>0$ a $t=t_N$ satisfying $t_N>N$ such that $\mathsf{E}|\zeta(t_N)|>\varepsilon$. Now choose some t_1. Since

$$\mathsf{E}|\zeta(t_1)|=\mathsf{E}|\xi(t_1)-\lim_{s\to\infty}\xi(t_1, s)|=\lim_{s\to\infty}\mathsf{E}|\xi(t_1)-\xi(t_1, s)|,$$

one can find s_1 such that $\mathsf{E}|\xi(t_1)-\xi(t_1, s_1)|>\varepsilon$. Set $t_2=t_1+s_1$ and choose $t_3>t_2$ such that $\mathsf{E}|\zeta(t_3)|>\varepsilon$. We continue this process indefinitely. Then

$$\mathsf{E}\sum_{1}^{2n-1}|\delta(t_k, t_{k+1})|\geq\mathsf{E}\sum_{1}^{n}|\delta(t_{2k-1}, t_{2k})|$$

$$=\mathsf{E}\sum_{1}^{n}|\xi(t_{2k})-\mathsf{E}\{\xi(t_{2k})|\mathfrak{F}_{t_{2k-1}}\}|\geq n\varepsilon\to\infty,$$

which contradicts the definition of a quasi-martingale.

Thus the existence of a decomposition satisfying the conditions of the theorem has been established We shall now prove its uniqueness

Let there be two decompositions $\xi(t)=\mu_1(t)-\zeta_1(t)=\mu_2(t)-\zeta_2(t)$ Then $\mu_1(t)-\mu_2(t)=\zeta_2(t)-\zeta_1(t)$, while $\mathsf{E}|\zeta_1(t)-\zeta_2(t)|\to 0$ as $t\to\infty$ On the other hand, $|\mu_1(t)-\mu_2(t)|$ is a submartingale and $\mathsf{E}|\mu_1(t)-\mu_2(t)|$ is a monotonically nondecreasing function of t Consequently, $\mathsf{E}|\mu_1(t)-\mu_2(t)|\equiv 0$ and $\mu_1(t)=\mu_2(t) \pmod{\mathsf{P}}$ Finally if $\xi(t)$ is a supermartingale, then

$$\mu(t)=\lim_{s\uparrow\infty}\xi(s,t)=\lim_{s\to\infty}\mathsf{E}\{\xi(s+t)\,|\,\mathfrak{F}_t\}\leq \xi(t),$$

which implies that $\zeta(t)=\xi(t)-\mu(t)\geq 0$ in this case □

Definition. A nonnegative supermartingale satisfying condition $\mathsf{E}\xi(t)\to 0$ as $t\to\infty$ is called a *potential*

Observe that for a potential the limit $\xi_\infty=\lim_{t\to\infty}\xi(t)$ exists and $\xi_\infty=0$ with probability 1

Corollary. *Supermartingale $\xi(t)$ satisfying the condition* $\inf \mathsf{E}\xi(t)>-\infty$ *admits the decomposition $\xi(t)=\mu(t)+\pi(t)$, where $\mu(t)$ is a martingale and $\pi(t)$ is a potential This decomposition is unique*

By analogy with the classical theory of superharmonic functions this decomposition is called the Riesz decomposition We shall agree to call the decomposition which was established in Theorem 4 Riesz's decomposition as well, and a quasi-martingale $\zeta(t)$ satisfying condition $\mathsf{E}|\zeta(t)|\to 0$ as $t\to\infty$ will be called *quasi-potential.*

We now show that an arbitrary quasi-potential can be represented as a difference of two potentials

Let $\zeta(t)$ be an arbitrary quasi-potential. Set

$$\delta_{k,n}=\delta\left(\frac{k}{2^n},\frac{k+1}{2^n}\right),$$

$$\delta_{k,n}^{+}=\max(\delta_{k,n},0),\qquad \delta_{k,n}^{-}=\delta_{k,n}^{+}-\delta_{k,n},$$

$$\pi_{+}^{n}(t)=\mathsf{E}\left\{\sum_{k=j(t)}^{\infty}\delta_{k,n}^{+}\,\middle|\,\mathfrak{F}_t\right\},\qquad \pi_{-}^{n}(t)=\mathsf{E}\left\{\sum_{k=j(t)}^{\infty}\delta_{k,n}^{-}\,\middle|\,\mathfrak{F}_t\right\},$$

where $j(t)$ is an integer defined by conditions $(j(t)-1)/2^n<t\leq j(t)/2^n$

Note that for $t=j/2^n$

$$\zeta(t)=\mathsf{E}\left\{\sum_{k=j}^{\infty}\delta_{k,n}\,\middle|\,\mathfrak{F}_t\right\}=\pi_{+}^{n}(t)-\pi_{-}^{n}(t),$$

and that the absolute convergence (mod P) of the series $\sum_{k=0}^{\infty}\delta_{k,n}$ and its integrability follow from the definition of a quasi-martingale Clearly $\mathsf{E}\{\pi_+^n(t)|\mathfrak{F}_s\} \le \pi_+^n(s)$ for $s<t$ so that $\pi_+^n(t)$ is a potential

We show that $\pi_+^n(t) \le \pi_+^{n+1}(t)$, $n=1,2,.$ Consider a summand appearing in the expression for $\pi_+^n(t)$, for example $\mathsf{E}\{\delta_{k\,n}^+|\mathfrak{F}_t\}$ $(k/2^n \ge t)$. We have

$$\begin{aligned}\mathsf{E}\{\delta_{k\,n}^+|\mathfrak{F}_t\} = \mathsf{E}\Big\{\Big[\zeta\Big(\frac{2k}{2^{n+1}}\Big) - \mathsf{E}\Big\{\zeta\Big(\frac{2k+1}{2^{n+1}}\Big)\Big|\mathfrak{F}_{\frac{2k}{2^{n+1}}}\Big\} \\ + \zeta\Big(\frac{2k+1}{2^{n+1}}\Big) - \mathsf{E}\Big\{\zeta\Big(\frac{2k+2}{2^{n+1}}\Big)\Big|\mathfrak{F}_{\frac{2k+1}{2^{n+1}}}\Big\}\Big|\mathfrak{F}_{\frac{k}{2^n}}\Big]^+\Big|\mathfrak{F}_t\Big\} \\ \le \mathsf{E}\{[\delta_{2k\,n+1}^+ + \delta_{2k+1\,n+1}^-]|\mathfrak{F}_t\}.\end{aligned}$$

This implies the monotonicity of the sequences $\pi_+^n(t)$ Moreover, it follows from that proven above that $\pi^n(t)$ is also a potential and that $\pi_-^n(t) \le \pi_-^{n+1}(t)$.

Now set

$$\pi_+(t) = \lim \pi_+^n(t), \qquad \pi_-(t) = \lim \pi_-^n(t)$$

These limits exist for each t with probability 1 Since $\mathsf{E}(\pi_+^n(t)+\pi_-^n(t)) = \mathsf{E}\sum_{k=0}^{\infty}|\delta_{k,n}| \le V$, it follows that $\mathsf{E}\pi_\pm(t)<\infty$. Clearly $\pi_\pm(t)$ are supermartingales

It is easy to verify that $\mathsf{E}\pi_\pm^n(t)\to 0$ as $t\to\infty$ uniformly in n Hence $\pi_\pm(t)$ are potentials

We now define processes $\pi_\pm(t)$ for all $t \ge 0$ in such a manner that their sample functions will be continuous from the right with probability 1. Taking into account the fact that the process $\zeta(t)$ is also continuous from the right, we observe that equality $\zeta(t) = \pi_+(t) - \pi_-(t)$ is valid for all $t \ge 0$ with probability 1.

Theorem 5. *If $\zeta(t)$ is a qu sipotential with sample functions belonging to $\mathscr{D}$, then potentials $\pi_+(t)$ and $\pi_-(t)$ exist such that with probability* 1

$$\zeta(t) = \pi_+(t) + \pi_-(t) \qquad \forall t \ge 0$$

Stopping and random time substitution. Now we shall consider submartingales

$$\{\xi(t), \mathfrak{F}_t, t \in T\},$$

where

$$T = N = \{0, 1, \ldots, n, \quad .\} \quad \text{or} \quad T = [0, \infty)$$

If $T=[0,\infty)$ we shall assume that the sample functions of the process $\xi(t)$ belong to $\mathscr{D}[0,\infty)$, $\mathfrak{F}_t = \mathfrak{F}_{t+}$, $t \in [0,\infty)$, and $\mathfrak{F}_0$ contains all the subsets of probability 0

Recall the definition of random time (Volume I, Chapter I, Section 1, Definition 14) Let $\{\mathfrak{F}_t, t \in T\}$ be a current of σ-algebras A function $\tau = f(\omega)$, $\omega \in \Omega_\tau \subset \Omega$, with values in T is called a *random time on* $\{\mathfrak{F}_t, t \in T\}$ or an $\mathfrak{F}_t$*-random time* if $\{\tau \le t\} \in \mathfrak{F}_t$ for all $t \in T$

In what follows we shall discuss only those random times which are defined on the whole space Ω(($\Omega_\tau = \Omega$)

A σ-algebra of events $\mathfrak{F}_\tau$ generated by events up to time τ which is called a σ-algebra corresponds to each random time τ This σ-algebra consists of events $B \in \mathfrak{S}$ such that

$$B \cap \{\tau \leq t\} \in \mathfrak{F}_t \qquad \forall t \in T$$

It is easy to verify that if $\tau_1 \leq \tau_2$ then $\mathfrak{F}_{\tau_1} \subset \mathfrak{F}_{\tau_2}$ (Volume I, Chapter I, Section 1) In Volume I, Chapter II, Section 2 (p 55) the following result was proved

Lemma 1. *Let T be a finite set, τ_k, $k = 1, \ldots, s$, be a sequence of random times on $\{\mathfrak{F}_t, t \in T\}$ defined on the whole space Ω and such that $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_s$, and let $\mathfrak{F}_k^* = \mathfrak{F}_{\tau_k}$ be a σ-algebra generated by the random time τ_k ($k = 1, \ldots, s$) If $\{\xi(t), \mathfrak{F}_t, t \in T\}$ is a supermartingale (martingale) then $\{\xi(\tau_k), \mathfrak{F}_k^*, k = 1, \ldots, s\}$ is also a supermartingale (martingale)*

We now generalize this result to the case of semimartingales considered in this section

Let $\{\xi(t), \mathfrak{F}_t, t \in T\}$ be a supermartingale satisfying the condition: an integrable random variable η exists such that

$$\xi(t) \geq E\{\eta \mid \mathfrak{F}_t\} \qquad (10)$$

Consider a random time τ taking on values in T and also possibly the value $t = \infty$ Set

$$\mathfrak{F}_\infty = \sigma\{\sigma(\eta), \mathfrak{F}_t, t \in [0, \infty)\},$$

$$\xi_\tau = \begin{cases} \xi(t) & \text{for } \tau = t, t \in T, \\ \eta & \text{for } \tau = \infty. \end{cases}$$

The random variable ξ_τ is $\mathfrak{F}_\tau$-measurable. Indeed, in the case of a discrete time, this assertion was proved earlier, in Volume I, Chapter I, Section 1, Lemma 5. For $T = [0, \infty)$ the proof is as follows.

Proof Introduce discrete approximations of the random time τ by setting $\tau^{(n)} = (k+1)/2^n$ if $\tau \in (k/2^n, (k+1)/2^n]$, $\tau^{(n)} = 0$ if $\tau = 0$. The continuity from the right of the sample functions of the process $\xi(t)$ implies that $\lim \xi(\tau_n) = \xi(\tau)$ On the other hand

$$\{\xi(\tau) < a\} \cap \{\tau < t\} = \lim \{\xi(\tau^{(n)}) < a\} \cap \{\tau^{(n)} < t\}$$

For $t \in (k/2^n, (k+1)/2^n]$ we have

$$\{\xi(\tau^{(n)}) < a\} \cap \{\tau^{(n)} < t\} \in \mathfrak{F}_{(k+1)/2^n}$$

Thus

$$\{\xi(\tau)<a\}\cap\{\tau<t\}\in\mathfrak{F}_{t+}=\mathfrak{F}_t.$$

Utilizing once again equality $\mathfrak{F}_{t+}=\mathfrak{F}_t$ we obtain

$$\{\xi(\tau)<a\}\cap\{\tau\leq t\}\in\mathfrak{F}_t$$

which implies that $\xi(\tau)$ is $\mathfrak{F}_\tau$-measurable. □

Theorem 6. *Let supermartingale $\xi(t)$ satisfy condition* (10), *let σ and τ be random times with $\sigma\leq\tau$ Then the variables ξ_σ and ξ_τ are integrable and*

$$\mathsf{E}\{\xi_\tau\,|\,\mathfrak{F}_\sigma\}\leq\xi_\sigma \tag{11}$$

Proof First consider the case $T=N$. Let $\sigma_k=\sigma\wedge k$ and $\tau_k=\tau\wedge k$ Set $\xi(t)=\zeta(t)+\eta(t)$, where $\eta(t)=\mathsf{E}\{\eta\,|\,\mathfrak{F}_t\}$, $\zeta(t)=\xi(t)-\eta(t)$. Conditions of the theorem imply that $\zeta(t)\geq 0$ and, moreover, $\zeta(t)$ is a supermartingale.

We shall start with the process $\zeta(t)$. Lemma 1 implies that $\mathsf{E}\zeta_{\tau_k}\leq\mathsf{E}\zeta_0$ Approaching the limit as $k\to\infty$ and utilizing Fatou's lemma we obtain

$$\mathsf{E}\zeta_\tau=\mathsf{E}\lim_{k\to\infty}\zeta_{\tau_k}\leq\mathsf{E}\zeta_0$$

so that $\mathsf{E}\zeta_\tau<\infty$

Now let $B=\mathfrak{F}_\sigma$. Then in view of Lemma 1

$$\int_{B\cap\{\tau\leq k\}}\zeta_\tau\,d\mathsf{P}\leq\int_{B\cap\{\sigma\leq k\}}\zeta_\tau\,d\mathsf{P}\leq\int_{B\cap\{\sigma\leq k\}}\zeta_{\sigma_k}\,d\mathsf{P}=\int_{B\cap\{\sigma\leq k\}}\zeta_\sigma\,d\mathsf{P}$$

Taking into account that $\zeta_\tau=\zeta_\sigma=0$ for $\sigma=\infty$ and approaching the limit as $k\to\infty$ in the obtained relationships, we have

$$\int_B\zeta_\tau\,d\mathsf{P}\leq\int_B\zeta_\sigma\,d\mathsf{P}; \tag{12}$$

this yields the assertion of the theorem for the process $\zeta(t)$.

We now consider process $\eta(t)$. This process is a uniformly integrable martingale. Observe that

$$\eta_\tau=\mathsf{E}\{\eta\,|\,\mathfrak{F}_t\}|_{t=\tau}=\mathsf{E}\{\eta\,|\,\mathfrak{F}_\tau\} \tag{13}$$

Indeed, if $A\in\mathfrak{F}_\tau$, $A_k=A\cap\{\tau=k\}$, $k=0,1,\quad.,n,\ldots,\infty$, then

$$\int_{A_k}\eta_\tau\,d\mathsf{P}=\int_{A_k}\mathsf{E}\{\eta\,|\,\mathfrak{F}_k\}\,d\mathsf{P}=\int_{A_k}\eta\,d\mathsf{P}.$$

Summing up these equalities over all the values of k, we obtain

$$\int_A\eta_\tau\,d\mathsf{P}=\int_A\eta\,d\mathsf{P}.$$