# Lecture Notes in Engineering

Edited by C. A. Brebbia and S. A. Orszag

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# The Boundary Integral Equation Method in Axisymmetric Stress Analysis Problems

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## FOREWORD

The Boundary Integral Equation (BIE) or the Boundary Element Method is now well established as an efficient and accurate numerical technique for engineering problems. This book presents the application of this technique to axisymmetric engineering problems, where the geometry and applied loads are symmetrical about an axis of rotation. Emphasis is placed on using isoparametric quadratic elements which exhibit excellent modelling capabilities. Efficient numerical integration schemes are also presented in detail.

Unlike the Finite Element Method (FEM), the BIE adaptation to axisymmetric problems is not a straightforward modification of the two- or three-dimensional formulations. Two approaches can be used; either a purely axisymmetric approach based on assuming a ring of load, or, alternatively, integrating the three-dimensional fundamental solution of a point load around the axis of rotational symmetry. Throughout this book, both approaches are used and are shown to arrive at identical solutions.

The book starts with axisymmetric potential problems and extends the formulation to elasticity, thermoelasticity, centrifugal and fracture mechanics problems. The accuracy of the formulation is demonstrated by solving several practical engineering problems and comparing the BIE solution to analytical or other numerical methods such as the FEM. This book provides a foundation for further research into axisymmetric problems, such as elastoplasticity, contact, time-dependent and creep problems.

I wish to express my sincere gratitude to Dr R.T. Fenner for his constant guidance, encouragement and excellent advice throughout the course of this work. I would also like to thank my colleagues; Drs K.H. Lee and E.M. Remzi for their valuable discussions on the BIE method, and Dr M.J. Abdul-Mihsein for his collaboration on Chapters 5 and 6. Thanks are also due to Mrs E.A. Hall for her skilful and accurate typing of this manuscript. Finally, I am indebted to my wife, Jane, for her patience and understanding throughout this work.

Stafford, England, December 1985 A.A. Bakr

## NOTATION

| | | |
|---|---|---|
| $A$ | : | area in a radial plane through the axis of rotational symmetry |
| $A$ | : | surface area of a crack |
| $[A]$ | : | matrix containing the integrals of the traction kernels |
| $A_{rr}, A_{rz}, A_{zr}, A_{zz}$ | : | coefficients of the sub-matrices of the matrix $[A]$ |
| $a_i$ | : | coefficients used to determine the elliptic integrals, $i = 1,5$ |
| $[B]$ | : | matrix containing the integrals of the displacement kernels |
| $B_{rr}, B_{rz}, B_{zr}, B_{zz}$ | : | coefficients of the sub-matrices of the matrix $[B]$ |
| $b_i$ | : | coefficients used to determine the elliptic integrals, $i = 1,5$ |
| $C$ | : | parameter contributing to the leading diagonal terms of the matrix $[A]$ in the potential problem |
| $[C]$ | : | solution matrix multiplying the unknown variables |
| $c_i$ | : | coefficients used to determine the elliptic integrals, $i = 1,5$ |
| $C_{rr}, C_{rz}, C_{zr}, C_{zz}$ | : | parameter contributing to the leading diagonal terms of the matrix $[A]$ in the elasticity problem |
| $[D]$ | : | matrix multiplying the known variables |
| $d(m,c)$ | : | number assigned to the $c$th node of the $m$th element |
| $d_i$ | : | coefficients used to determine the elliptic integrals, $i = 1,5$ |
| $E$ | : | Young's modulus |
| $[E]$ | : | matrix containing the known coefficients to be solved in the potential and elasticity problems |
| $[E']$ | : | matrix containing the known coefficients to be solved in the thermoelasticity problem |
| $E(m,\frac{\pi}{2})$ | : | complete elliptic integral of the second kind of modulus $m$ |
| $e$ | : | percentage compression of a rubber block |
| $\underline{e}_r, \underline{e}_z$ | : | unit vectors in the radial and axial directions |
| $e_{rr}, e_{zz}, e_{\theta\theta}$ | : | strains in the radial, axial and hoop directions |
| $e_{rz}$ | : | shear strain |
| $\underline{F}$ | : | body force vector |
| $[F]$ | : | matrix containing the integrals of the thermoelastic kernels multiplying the temperatures |
| $F_r, F_z$ | : | components of the body force vector in the radial and axial directions |
| $f$ | : | function to be integrated using the ordinary Gaussian quadrature technique |
| $f^*$ | : | modified function to be integrated using the logarithmic Gaussian quadrature technique |

| | | |
|---|---|---|
| $G$ | : | total number of Gaussian quadrature points |
| $\underline{G}$ | : | Galerkin vector |
| $[G]$ | : | matrix containing the integrals of the thermoelastic kernels multiplying the temperature gradients |
| $[G']$ | : | matrix containing the known coefficients to be solved in the centrifugal problem |
| $G_r$ , $G_z$ | : | components of the Galerkin vector in the radial and axial directions |
| $G_I$ , $G_{II}$ , $G_{III}$ | : | strain energy release rate for fracture modes I, II and III |
| $H$ | : | height of a cylinder |
| $H_c$ | : | functions remaining non-zero over the range of integration |
| $H_n$ | : | Hankel transform of order $n$ |
| $h$ | : | ratio between the heat transfer coefficient to the thermal conductivity |
| $I_r$ , $I_z$ | : | integrals of the thermoelastic kernels in the radial and axial directions |
| $J$ | : | Jacobian of transformation |
| $J$ | : | $J$-contour integral |
| $J_n$ | : | Bessel function of order $n$ |
| $J_r$ , $J_z$ | : | components of the Jacobian of transformation in the radial and axial directions |
| $K(m,\frac{\pi}{2})$ | : | complete elliptic integral of the first kind of modulus $m$ |
| $K_1$ | : | first potential kernel multiplying the potential gradient |
| $K_2$ | : | second potential kernel multiplying the potential gradient |
| $\underline{K}_I$ | : | normalised stress intensity factor |
| $K_I$ , $K_{II}$ , $K_{III}$ | : | stress intensity factors for fracture modes I, II and III |
| $K_{c1}$ , $K_{c2}$ | : | axisymmetric centrifugal kernels |
| $K_{r1}$ , $K_{r2}$ | : | axisymmetric thermoelastic kernels in the radial direction |
| $K_{z1}$ , $K_{z2}$ | : | axisymmetric thermoelastic kernels in the axial direction |
| $k$ | : | thermal conductivity |
| $M$ | : | total number of nodes |
| $m$ | : | modulus of the elliptic integrals |
| $m_{1r}$ , $m_{1z}$ | : | components of the unit tangential vector in the radial and axial directions |
| $N_c$ | : | shape function associated with a nodal point c |
| $\underline{n}$ | : | unit outward normal to the surface $S$ |
| $n_r$ , $n_z$ | : | components of the unit outward normal in the radial and axial directions |

| | | |
|---|---|---|
| $P$ | : | arbitrary boundary point |
| $p$ | : | load point inside the solution domain |
| $p_r, p_z$ | : | components of the ring load vector at $p$ in the radial and axial directions |
| $Q$ | : | field boundary point |
| $Q_{n-\frac{1}{2}}$ | : | Legendre function of the second kind of order zero and degree $n-\frac{1}{2}$ |
| $q$ | : | interior point in the volume $V$ |
| $R$ | : | radial distance measured from the centre of a sphere |
| $R_1$ | : | inner radius of a cylinder or sphere |
| $R_2$ | : | outer radius of a cylinder or sphere |
| $R_0$ | : | radius of a round bar or solid cylinder |
| $R_p$ | : | fixed radial coordinate of the load point $p$ |
| $r(p,Q)$ | : | physical distance between points $p$ and $Q$ |
| $r_Q$ | : | variable radial coordinate of the boundary point $Q$ |
| $r_q$ | : | variable radial coordinate of the interior point $q$ |
| $S$ | : | surface of the volume $V$ |
| $S'$ | : | distance on the path $\Gamma'$ |
| $S_\varepsilon$ | : | surface of the sphere of radius $\varepsilon$ |
| $s$ | : | arbitrary scalar quantity |
| $T_1, T_2$ | : | temperatures at the internal and external surfaces of a cylinder or sphere |
| $T_{ij}$ | : | traction kernel functions in Cartesian coordinates, $i = 1,3$, $j = 1,3$ |
| $T_{rr}, T_{rz}, T_{zr}, T_{zz}$ | : | axisymmetric traction kernel functions |
| $t_1, t_2$ | : | tractions in the directions tangential and normal to the surface |
| $t_r, t_z$ | : | components of the traction vector in the radial and axial directions |
| $U$ | : | strain energy of the body |
| $U_{ij}$ | : | displacement kernel functions in Cartesian coordinates, $i = 1,3$, $j = 1,3$ |
| $U_{rr}, U_{rz}, U_{zr}, U_{zz}$ | : | axisymmetric displacement kernel functions |
| $\underline{u}$ | : | displacement vector |
| $u_R$ | : | displacement in the radial direction from the centre of a sphere |
| $u_r, u_z$ | : | components of the displacement vector in the radial and axial directions |
| $V$ | : | volume of the solution domain |
| $V_\varepsilon$ | : | volume of the sphere of radius $\varepsilon$ |
| $\underline{v}$ | : | arbitrary vector quantity |
| $v_r, v_z$ | : | components of an arbitrary vector in the radial and axial directions |

| | | |
|---|---|---|
| $W$ | : | strain energy density of the body |
| $w_g$ | : | weighting functions associated with ordinary Gaussian quadrature points |
| $w^*_g$ | : | weighting functions associated with logarithmic Gaussian quadrature points |
| $X_p$ | : | fixed x-coordinate of the load point $p$ |
| $[x]$ | : | vector of unknown quantities |
| $x_Q$ | : | variable x-coordinate of the boundary point $Q$ |
| $Y_p$ | : | fixed $y$-coordinate of the load point $p$ |
| $[y]$ | : | vector of unknown quantities |
| $y_Q$ | : | variable $y$-coordinate of the boundary point $Q$ |
| $Z_p$ | : | fixed axial coordinate of the load point $p$ |
| $z_Q$ | : | variable axial coordinate of the field point $Q$ |
| $z_q$ | : | variable axial coordinate of the interior point $q$ |
| $\alpha$ | : | coefficient of thermal expansion |
| $\Gamma$ | : | surface path in any radial plane through the axis of rotational symmetry |
| $\Gamma'$ | : | path from one surface of the crack to the other inside the solution domain |
| $\Gamma_c$ | : | common interface between two subdomains |
| $\gamma$ | : | parameter of Legendre functions of the second kind |
| $\gamma$ | : | specific surface energy of the body |
| $\delta$ | : | Dirac delta function |
| $\delta_{ij}$ | : | Kronecker delta |
| $\varepsilon$ | : | radius of small sphere centred at the load point $p$ |
| $\theta_p$ | : | angular coordinate of the load point $p$ |
| $\theta_Q$ | : | angular coordinate of the boundary point $Q$ |
| $\mu$ | : | shear modulus |
| $\nu$ | : | Poisson's ratio |
| $\xi$ | : | local or intrinsic coordinate |
| $\rho$ | : | density of the material |
| $\rho$ | : | distance from crack tip |
| $\sigma_1, \sigma_2, \sigma_3$ | : | principal stresses |
| $\sigma_{11}, \sigma_{22}, \sigma_{33}$ | : | direct stresses in local directions 1, 2 and 3 |
| $\sigma_{12}, \sigma_{21}$ | : | shear stresses in the local directions 1 and 2 |
| $\sigma_c$ | : | critical stress required for crack growth |
| $\sigma_e$ | : | von Mises equivalent stress |
| $\sigma_{nett}$ | : | nett stress acting on the cross-section at the crack plane |
| $\sigma_R$ | : | direct stress in the radial direction from the centre of a sphere |

| | | |
|---|---|---|
| $\sigma_{rr}$ , $\sigma_{zz}$ , $\sigma_{\theta\theta}$ | : | direct stresses in the radial, axial and hoop directions |
| $\sigma_{rz}$ | : | shear stress |
| $\sigma_{t}$ | : | direct stress in the tangential direction to the surfaces of a sphere |
| $\phi$ | : | unknown harmonic function satisfying Laplace's equation |
| $\phi_1$ , $\phi_2$ | : | potentials at the inner and outer surfaces of a cylinder or sphere |
| $\psi$ | : | potential function |
| $\psi$ | : | fundamental solution for Laplace's equation in three-dimensional Cartesian coordinates |
| $\omega$ | : | angular velocity |

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## TABLE OF CONTENTS

# CHAPTER 1
# INTRODUCTION AND AIMS

## 1.1 INTRODUCTION

In many engineering problems, both the geometry and the applied boundary conditions are symmetrical about some axis. Axisymmetric geometries, or bodies of revolution, are formed by rotating a two-dimensional plane through 360° about an axis. Therefore, such bodies can be represented by a typical radial plane passing through the axis of rotational symmetry, subsequently reducing the analysis from three dimensions to two dimensions; the radial and axial directions. Axisymmetric engineering problems arise, for example, in pressure vessels, nuclear reactors, pipes, mechanical seals and shafts.

Analytical solutions in engineering are often limited to simple geometries and loading. In complex practical engineering problems, they rarely yield acceptable results for design purposes, and at best produce results which are unreasonably approximated, resulting in the use of rather large factors of safety. Experimental techniques, on the other hand, can produce accurate results but tend to be very expensive and time-consuming. The advent of high speed digital computers made it often more accurate and economical to use numerical methods in engineering. Indeed, computational methods are now an inseparable part of engineering design, manufacture, research and development.

The Finite Element Method (FEM) is perhaps the most widely used numerical technique in engineering. This method, based on dividing the solution domain into small segments or elements, has proved very versatile, and is constantly being developed and improved. However, there are some drawbacks in using the FEM. One of the drawbacks is the need to use a large amount of small elements in regions of high stress concentration, and the computing cost associated with solving a large set of simultaneous equations is often very high, although the coefficient matrix is symmetric and of banded form. Also, most finite element analyses give less accurate results for stresses than for displacements; the former being of more practical use to engineers. Perhaps the most serious drawback in using the FEM to solve complex engineering problems is that the data input for mesh geometry and loading conditions can be a very tiresome, time-consuming and error-prone process, even when using sophisticated mesh generation programs. Furthermore, the FEM programs yield a large amount of output data, most of which remain unused, particularly at internal points.

Recently, a new numerical technique called the Boundary Integral Equation (BIE) method, or the Boundary Element Method (BEM), was initially developed for potential problems and later extended to elasticity problems due to the analogy between potential theory and classical elasticity theory. This method discretises only the boundary of the problem and not the whole solution domain, resulting in the reduction of dimensionality by one. Thus, three-dimensional geometries are modelled by two-dimensional elements, while axisymmetric and two-dimensional geometries are modelled by one-dimensional line elements. This is the main advantage of the BIE method over the FEM since the time taken to input and interpret the data for a particular problem is substantially reduced. This advantage is particularly important in modern day analysis of engineering problems, because computers are becoming so efficient and commercially inexpensive that the cost of the engineer's or analyst's time spent on the data input is more significant than the cost of the computer time used in performing the analysis. Also, in practical applications, the BIE method often needs less computer time and core storage than the FEM.

Although in most engineering applications the regions of interest are on the boundary of the problem, it may be desired to obtain results at interior points inside the solution domain. Because there is no further approximation in modelling the interior behaviour, the BIE method produces more accurate solutions at these points than the FEM, with the ability to concentrate only on the region of interest, resulting in a more efficient use of computer resources. Further, the solution of nearly or exactly incompressible material problems presents serious difficulties and errors when using the conventional displacement-based FEM, because the general stress-strain equations of elasticity contain terms that become infinite as Poisson's ratio reaches 0.5, while the BIE method accommodates such problems without any difficulty due to the nature of the integral equations used in the analysis.

Unlike the FEM, where the axisymmetric analysis is performed by simply modifying the two-dimensional analysis, the BIE adaptation for axisymmetric problems is far from being a straightforward modification of the two- or three-dimensional analysis. There are two alternative approaches to arrive at the axisymmetric form of the BIE method. The first is to integrate the three-dimensional fundamental solution in a circular path around the axis of rotational symmetry, while the second assumes from the outset axisymmetric fundamental solutions. In this book, both approaches are used and shown to arrive at identical solutions. The boundary is discretised into isoparametric quadratic elements, where both the geometry and variables are allowed to vary quadratically over the boundary. These elements offer excellent modelling capabilities, particularly in complex geometries and stress concentration or crack problems. Due to the quadratic shape functions used in these elements, analytical integrations become impractical, and all integrations are performed numerically using the Gaussian quadrature technique.

## 1.2 LITERATURE SURVEY - AXISYMMETRIC PROBLEMS

In 1963, Jaswon [1] and Symm [2] implemented the BIE method for two-dimensional potential problems, using straight line elements to discretise the boundary and with the unknowns assumed constant over each element. Rizzo [3] and Cruse [4] extended the technique to two- and three-dimensional problems, respectively, using straight line and flat triangular boundary elements with variables assumed to be constant over each element. Improvements followed which allowed the variables to change linearly [5,6], and then both the geometry and variables to change quadratically over each element [7,8].

The first applications of the BIE method to axisymmetric elasticity were first effected by Kermanidis [9], Mayr [10] and Cruse, et al. [11], using either constant or linear variation elements, sometimes in the form of circular arcs. Further developments in elasticity included the use of isoparametric quadratic elements by Bakr [12] and Bakr & Fenner [13,14], and the treatment of axisymmetric geometries subject to arbitrary boundary conditions by Rizzo & Shippy [15], Rizzo, et al. [16], Nigam [17] and Mayr, et al. [18]. Several researchers applied the BIE method to axisymmetric potential problems, including Wrobel & Brebbia [19], Gupta [20], Shippy, et al. [21], Bakr & Fenner [22], Wrobel [23], Yoshikawa & Tanaka [24] and Au & Brebbia [25]. Other applications in axisymmetric problems included elastoplasticity [26,27], viscoplasticity [28], fracture mechanics [29], pressure vessels [30] and body force loading [31,32]. The axisymmetric formulation has also been included in some recent BIE books (see, for example, reference [33]). This recent extension of the BIE method to axisymmetric problems emphasises the importance of such problems, and the need for accurate solutions without resorting to three-dimensional BIE analysis.

## 1.3 LAYOUT OF NOTES

This book begins with the application of the BIE method to axisymmetric potential problems, and then extends the formulation to axi-