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# Field Analysis and Electromagnetics

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FIELD ANALYSIS AND ELECTROMAGNETICS

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# *Preface*

This text contains sufficient material for a three-term, three-hour-per-week course. It is, however, primarily intended for a two-term undergraduate course in which the individual instructor may choose from among a variety of optional topics for detailed treatment. If the student has had prior training in advanced calculus (or if the text is used for graduate studies), it may be possible to treat the first four chapters as review material, with emphasis placed on the physical interpretation of mathematical relations. Such a review will also serve the purpose of familiarizing the reader with the notation introduced in these chapters and used throughout the text.

Depending on the requirement of the undergraduate curriculum, many sections and some of the proofs may be omitted to provide shorter courses. These sections and proofs are marked, and their omission does not break the continuity of the text. In most cases we have first stated what is to be proved before proceeding to prove it. This makes it possible to use the text with varying degrees of mathematical involvement. The authors have found that the availability of detailed derivation in the text makes it possible to use the classroom hours for a discussion of the physical significance of the results and for problem solving, rather than for a reproduction of the derivations.

The problems given at the end of the text are designed to complement and extend the text material. For this reason they are numbered by chapter, section, and problem. For example, Prob. 1-4.3 deals with material discussed in Sec. 1-4 and is the third such problem. This numbering will assist the reader in locating the pertinent discussion. Throughout the text the rationalized mks-coulomb system of units is used and, for brevity, units of various quantities are not always explicitly stated.

The principal motivation for writing this text is best understood in terms of our objections to some of the currently available treatments of the subject. Although many undergraduate texts in electromagnetics have been published during the past decade, most of these are written with the idea that field theory is really graduate material and that there is no need to deal with it in depth at the undergraduate level. The courses based on such texts include only that mathematical preparation which is essential to solve the problems at hand; they do not provide the student with reasonable preparation for graduate studies. In accordance with this philosophy, the laws of electromagnetic theory are introduced piecemeal, so that the student does not learn a law in any greater generality than is necessary to solve the next set of problems.

Because of the lack of depth in such courses, most graduate schools have to start their field analysis and electromagnetic theory from "scratch," ignoring the undergraduate training of students. In fact, it is not an unusual experience to find such a training a hindrance rather than a help to graduate studies, inasmuch as some of the oversimplified concepts learned in undergraduate studies must be unlearned before more exact concepts can replace them. Furthermore, the low level of analysis currently prevalent in undergraduate courses in electromagnetic theory is inconsistent with the fact that in many schools advanced calculus is now taught to engineering students.

Notwithstanding this background in advanced calculus, it has been our experience that the formal approach of mathematics courses does not adequately prepare the student for the application of this material to physical problems. For this reason we have found it necessary to devote the first five chapters of the text to field analysis, in which we introduce the concepts of divergence, curl, and laplacian, not as mere differential operators, but as volume densities of the sources of a field. Similarly, a uniqueness theorem is treated not only as a mathematical structure but also as an expression of what constitutes physical determinism. We have found this approach most effective with students whether or not they have taken a course in advanced calculus.

In addition to the preparation for the study of electromagnetics, the first five chapters provide a self-contained and sound basis for the mathematical analysis of all kinds of fields and should prove valuable to physicists and mechanical, chemical, and civil engineers as well as electrical engineers.

With the introduction of the general concepts of fields and their sources, it is possible to state the laws of electromagnetic theory in an integrated manner. This is done in Chapter 6 where, in addition to Maxwell's equations, the concepts of energy conservation (Poynting's theorem) and electromagnetic momentum, as well as electromagnetic waves and potentials, are introduced. This chapter is a complete statement of the classical laws of electromagnetic theory, and the rest of the text is concerned with developing the ramifications of the theory and with its application to practical problems. Chapter 7 is devoted to introducing the technique of multipole expansion and the use of electric and magnetic multipole approximations. Chapter 8 deals, in detail, with the consequences of idealizations made in practice, where distributed charges and currents are assumed to be singular in the form of surface or line densities or point sources. This and the assumption of abrupt boundaries between two media lead to (nonphysical) discontinuities in the electric and magnetic fields at the idealized boundaries. The chapter shows how these idealizations are used to solve practical problems. The use of the Green's function and the concepts of coefficients of inductance and capacitance are also introduced in this chapter. Chapter 9 contains a detailed discussion of the method of separation of variables as applied to the solution of boundary-value problems. Also included in this chapter are numerical and graphical methods for solving problems whose boundaries do not permit direct analysis. Some microscopic properties of matter are considered in Chapter 10. In Chapter 11, radiating fields are introduced by means of a discussion of the elementary dipole antenna. Radiating fields are then contrasted with quasi-static fields—leading to a discussion of the field basis of circuit theory. Chapter 12 contains a comprehensive analysis of plane waves and applies this analysis to the study of transmission lines and waveguides. The general relationships between the transverse and longitudinal components of a plane wave derived in this chapter provide the basis for the study of guides of arbitrary cross section, although only rectangular guides are treated in detail.

Chapter 13 is an introduction to propagation in lossy media; it ends with a discussion of dispersion due to lossiness of the medium as well as geometric and parametric dispersion. Chapter 14 deals with the reflection of plane waves and the exchange of momentum between an electromagnetic field and a material medium. Chapter 15 uses the formulas derived in Chapter 11 as a basis for analyzing radiation from linear and loop antennas. This final chapter concludes with a discussion of antenna arrays and receiving antennas.

The material of this text has been taught by the authors and some of their colleagues from notes for the last three years at The City College of New York.

The authors wish to thank Professor John Truxal, Vice President in charge of Education at the Polytechnic Institute of Brooklyn, for his encouragement and help throughout the preparation of the manuscript. Thanks are also due to our Department Chairman, Professor Herbert Taub, for his cooperation. The authors have benefited from long discussions with many of our colleagues—most particularly, with Professor Egon Brenner of The City College and Professor Leonard Bergstein of the Polytechnic Institute of Brooklyn. Words cannot express the depth of our gratitude to Miss Sadie Silverstein, without whose cooperation in preparing the manuscript this text could not have been published. We are also grateful to our former student, Miss Rosalind Soodak, for drawing most of the original figures. Finally, we wish to thank our students who have suffered through early drafts and have provided us with innumerable corrections.

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# 1†

# *Vector Algebra*

In the process of learning and through a comparison of objects, the positive real integers become familiar concepts. Other types of numbers are subsequently introduced as abstractions to facilitate the analysis of observed phenomena. Fractional, negative, and irrational numbers are familiar examples. In the study of physical phenomena one encounters a variety of quantities. A quantity whose only measurable attribute is its magnitude is called a scalar quantity. The price of a given commodity and the temperature at a given location are examples of scalar quantities. A quantity with the two measurable attributes magnitude and direction is referred to as a vector quantity. The velocity of a moving object and the force applied to a body are examples of vector quantities. Thus a vector quantity can have the dimension of force or velocity or other dimensions.

In specifying a vector quantity it is customary to indicate it by a directed line (arrow) in the direction of the vector quantity. The length of the line is often made proportional to the magnitude of the quantity. Such a line is referred to as a vector. In this text we designate a vector by a boldface letter,‡ such as **u** or **A**. A dimensionless vector whose magnitude is unity is called a unit vector.

Two vectors having the same magnitude and direction are called equal vectors. Two equal vectors may be associated with the same point or different points in space.

**1-1. Vector and Scalar Algebra.** The above description of a vector is based on the concept of quantities with magnitude and direction. Mathematics deals with numbers which are abstractions, not with quantities. The numbers 2 and 5 are abstractions based on consideration of, for example, two hands and five fingers. Similarly, vector numbers are abstractions based on quantities with two measurable attributes, magnitude and direction.

In making abstractions the intuitive ground of comparison between objects, which introduced us to integers, is lost. Instead, numbers are defined through their rules of operation. For example, the rules of addition and multiplication in common algebra define the algebraic numbers.

A sufficient set of rules for common algebra is:

1. $a + b = b + a$, commutation in addition.
2. $a + (b + c) = (a + b) + c$, association in addition.
3. There is an $x$ such that $a + x = b$, unique definition of subtraction.
4. $a \cdot b = b \cdot a$, commutation in multiplication.
5. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, association in multiplication.

† This chapter may be omitted by the reader acquainted with vector algebra.
‡ The magnitude of the vector **A** will be designated by the italic letter $A$.

6. If $a \neq 0$, there is an $x$ such that $a \cdot x = b$, unique definition of division.
7. If $a \cdot b = 0$, then $a = 0$ and/or $b = 0$.
8. $a \cdot (b + c) = a \cdot b + a \cdot c$, distributive property of addition and multiplication.

These rules are postulated to define the common algebra of scalars. However, they are chosen to make the algebra of scalars useful in describing phenomena. In order to provide an algebra that is useful in describing a broader field of phenomena, it will be found necessary to weaken the restrictions placed on the operations defining common algebra. This process is called "weakening the postulates." In the formulation of a useful algebra of vectors, postulates 4, 5, 6, and 7 mentioned above must be abridged or completely eliminated.

**1-2. Addition of Vectors.** Two vectors **A** and **B** which are not in the same or opposite directions determine a plane. The sum of two vectors **A** and **B** is a third

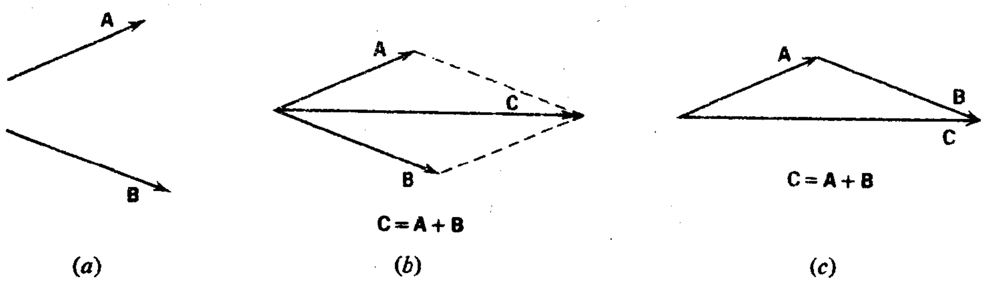


FIG. 1-1 Vector addition. (*a*) Two vectors; (*b*) parallelogram rule; (*c*) head-to-tail rule.

vector **C** in the plane of **A** and **B** and is obtained by the "parallelogram" rule. This rule states that to obtain the vector $\mathbf{C} = \mathbf{A} + \mathbf{B}$ two vectors must be drawn from a point, one equal to **A** and the other equal to **B**, as in Fig. 1-1*b*. Then a parallelogram with these two vectors as its sides should be drawn. The diagonal of this parallelogram, directed as shown in Fig. 1-1*b*, is the vector sum of **A** and **B**. This process is, of course, equivalent to the "head-to-tail" process of addition of vectors shown in Fig. 1-1*c*. The above definition of addition is based on the observation of effects caused by vector quantities such as forces and velocities.

The parallelogram rule is an operational definition of vector addition and implies the following rules:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \qquad \text{commutation in addition}$$

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C} \qquad \text{association in addition}$$

Furthermore, it implies that for two given vectors **A** and **B** there is always a third vector **X** such that $\mathbf{A} + \mathbf{X} = \mathbf{B}$. This vector is called the vector difference between **B** and **A**. Thus all the addition rules of common algebra are carried over into vector algebra.

**1-3. The Product of a Vector and a Scalar; Unit Vectors.** The product of a scalar $m$ with a vector $\mathbf{u}$ is $\mathbf{p} = m\mathbf{u}$, a vector in the direction of $\mathbf{u}$ with magnitude $mu$. Any vector **B** may be expressed as the product of its magnitude $B$ and a unit vector $\mathbf{a}_B = \mathbf{B}/B$; thus $\mathbf{B} = \mathbf{a}_B B$.

**1-4. Vector Representation of Plane and Elemental Surfaces.** With any plane surface we associate a vector **S** of magnitude equal to the area of the surface and in a direction normal to the surface. We refer to the direction of the normal as the direction of the vector area **S**. If the plane surface forms part of a closed surface (one which completely encloses a volume), we define the outward normal as its positive direction. Otherwise, the positive sense of S (the positive direction of the vector's arrowhead) is arbitrarily chosen.

Any infinitesimal portion $da$ of a curved surface may be considered planar. An infinitesimal vector $d\mathbf{a}$ is associated with this infinitesimal plane so that $d\mathbf{a} = \mathbf{n}\, da$, where $\mathbf{n}$ is the unit vector normal to the infinitesimal (and therefore planar) surface at the point under consideration.

**1-5. The Scalar Product of Two Vectors.** The scalar product of two vectors **A** and **B**, designated by the symbol $\mathbf{A} \cdot \mathbf{B}$, is defined as

$$\mathbf{A} \cdot \mathbf{B} \triangleq AB \cos \theta_{AB} \tag{1-1}$$

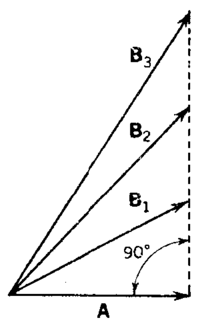


FIG. 1-2 Illustration of $\mathbf{B}_1 \cdot \mathbf{A} = \mathbf{B}_2 \cdot \mathbf{A}$ when $\mathbf{B}_1 \neq \mathbf{B}_2$.

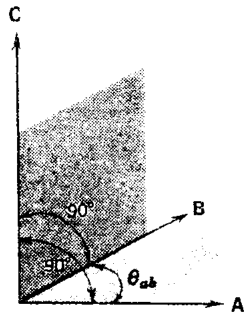


FIG. 1-3 Illustration of $\mathbf{C} = \mathbf{A} \times \mathbf{B}$.

where $\theta_{AB}$ is the angle between the vectors **A** and **B**.† The definition given in Eq. (1-1) implies the following laws:

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \qquad \text{commutative law for scalar product}$$

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \qquad \text{distributive law for scalar product}$$

Note that the specification of A and $\mathbf{A} \cdot \mathbf{B}$ does not uniquely define **B**. For example, assume that in Fig. 1-2 the vector A is of unit magnitude and that it is required to determine **B** so that $\mathbf{A} \cdot \mathbf{B} = 1$. It is apparent that the vectors $\mathbf{B}_1$, $\mathbf{B}_2$, and $\mathbf{B}_3$ are only three of an infinite number of vectors satisfying this requirement. If we consider the operation defined by Eq. (1-1) as multiplication, then the indeterminacy of **B** in the above example indicates that "division" by a vector is not defined. It is also noted that $\mathbf{A} \cdot \mathbf{B} = 0$ does not necessarily imply $\mathbf{A} = 0$ or $\mathbf{B} = 0$ but might rather imply that **A** and **B** are at right angles. The scalar product $\mathbf{A} \cdot \mathbf{B}$ is commonly referred to as the dot product.

**1-6. The Vector Product of Two Vectors.** The vector product of the two vectors **A** and **B** is designated as $\mathbf{A} \times \mathbf{B}$. It is defined to be the vector C of magnitude $AB \sin \theta_{AB}$ and in a direction normal to both **A** and **B** (see Fig. 1-3). The sense of **C** is determined by the "right-hand rule." This rule states that if a right-handed screw is turned from **A** toward **B** (in the direction of the smaller angle between A and **B**), then the sense of the arrow representing $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ is in the direction of advance of the screw. The above definition of the vector product implies the following:

† The symbol $\triangleq$ is read "is by definition equal to."

The anticommutative law holds for vector products:

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

Thus

$$\mathbf{A} \times \mathbf{A} = \mathbf{0}$$

The distributive law holds for vector products:

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$$

The associative law does not hold for vector products:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$$

This is apparent when we observe that the left-hand member must be normal to **A**, while the right-hand member is normal to **C**. From the definition of $\mathbf{A} \times \mathbf{B}$ it is also seen that $\mathbf{A} \times \mathbf{B} = \mathbf{0}$ implies $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$ or $\sin \theta_{AB} = 0$. Also, as with the dot product, the specification of **A** and $\mathbf{A} \times \mathbf{B}$ does not uniquely determine **B**. The vector product $\mathbf{A} \times \mathbf{B}$ is commonly referred to as the cross product.

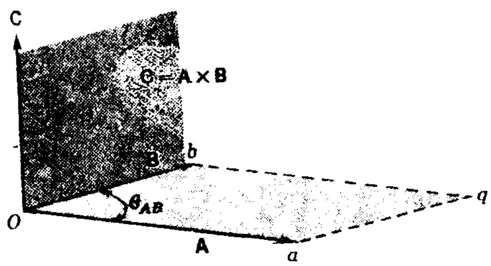


FIG. 1-4 The sense of vector **C** corresponds to the sense of the periphery *oaqb*.

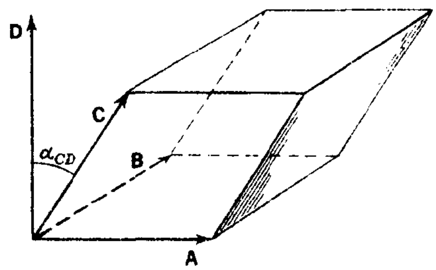


FIG. 1-5 Triple scalar product interpreted as a volume.

In the expression $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ the vectors **A** and **B** form the adjacent sides of a parallelogram in the plane of **A** and **B**, as shown in Fig. 1-4. The area of this parallelogram is $AB \sin \theta_{AB}$. Thus **C** is the vector area of this parallelogram. This is one (rather restricted) way of defining the sense of a vector area which does not form part of a closed surface. A more general way is to relate the sense of the vector area to the description of its periphery by the right-hand rule. Thus the sense of the vector **C** corresponds to a description of the periphery as being directed from $o$ to $a$ to $q$ to $b$ and back to $o$ (see Fig. 1-4).

**1-7. Triple Scalar Product.** The product $\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$ is a scalar. Its value is the volume of the parallelepiped formed by the vectors **A**, **B**, and **C** (see Fig. 1-5). The proof of this statement is seen by considering $\mathbf{A} \times \mathbf{B} = \mathbf{D}$. From Fig. 1-5

$$\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{C} \cdot \mathbf{D} = DC \cos \alpha_{CD}$$

From this figure it is seen that **D** is the area of the base of the parallelepiped and $C \cos \alpha_{CD}$ is the height of the volume. The triple product $\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$ is usually written as $\mathbf{C} \cdot \mathbf{A} \times \mathbf{B}$. Because of the nature of the operations no ambiguity will appear if the parentheses are removed. For example, $\mathbf{C} \cdot \mathbf{A} \times \mathbf{B}$ cannot mean $(\mathbf{C} \cdot \mathbf{A}) \times \mathbf{B}$ since $\mathbf{C} \cdot \mathbf{A}$ is a scalar and the vector product is defined only for two vectors.

We note that the volume of the parallelepiped defined by three (noncoplanar) vectors **A**, **B**, and **C** can be equally expressed as $\mathbf{C} \cdot \mathbf{A} \times \mathbf{B}$ or $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$ or any of four other possible combinations of the vectors **A**, **B**, and **C** and the dot and cross