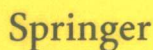


Derivatives and Geometry in $\mathbb{R}^3$



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Applied Mathematics: Body and Soul

[VOLUME 1]

Derivatives and Geometry in $\mathbb{R}^3$



Springer

Preface

I admit that each and every thing remains in its state until there is reason for change. (Leibniz)

The Need of Reform of Mathematics Education

Mathematics education needs to be reformed as we now pass into the new millennium. We share this conviction with a rapidly increasing number of researchers and teachers of both mathematics and topics of science and engineering based on mathematical modeling. The reason is of course the computer revolution, which has fundamentally changed the possibilities of using mathematical and computational techniques for modeling, simulation and control of real phenomena. New products and systems may be developed and tested through computer simulation on time scales and at costs which are orders of magnitude smaller than those using traditional techniques based on extensive laboratory testing, hand calculations and trial and error.

At the heart of the new simulation techniques lie the new fields of Computational Mathematical Modeling (CMM), including Computational Mechanics, Physics, Fluid Dynamics, Electromagnetics and Chemistry, all based on solving systems of differential equations using computers, combined with geometric modeling/Computer Aided Design (CAD). Computational modeling is also finding revolutionary new applications in biology, medicine, environmental sciences, economy and financial markets.

Education in mathematics forms the basis of science and engineering education from undergraduate to graduate level, because engineering and science are largely based on mathematical modeling. The level and the quality of mathematics education sets the level of the education as a whole. The new technology of CMM/CAD crosses borders between traditional engineering disciplines and schools, and drives strong forces to modernize engineering education in both content and form from basic to graduate level.

Our Reform Program

Our own reform work started some 20 years ago in courses in CMM at advanced undergraduate level, and has through the years successively penetrated through the system to the basic education in calculus and linear algebra. Our aim has become to develop a complete program for mathematics education in science and engineering from basic undergraduate to graduate education. As of now our program contains the series of books:

1. *Computational Differential Equations*, (*CDE*)
2. *Applied Mathematics: Body & Soul I-III*, (*AM I-III*)
3. *Applied Mathematics: Body & Soul IV-*, (*AM IV-*).

AM I-III is the present book in three volumes I-III covering the basics of calculus and linear algebra. *AM IV-* offers a continuation with a series of volumes dedicated to specific areas of applications such as *Dynamical Systems (IV)*, *Fluid Mechanics (V)*, *Solid Mechanics (VI)* and *Electromagnetics (VII)*, which will start appearing in 2003. *CDE* published in 1996 may be viewed as a first version of the whole *Applied Mathematics: Body & Soul* project.

Our program also contains a variety of software (collected in the *Mathematics Laboratory*), and complementary material with step-by step instructions for self-study, problems with solutions, and projects, all freely available on-line from the web site of the book. Our ambition is to offer a “box” containing a set of books, software and additional instructional material, which can serve as a basis for a full applied mathematics program in science and engineering from basic to graduate level. Of course, we hope this to be an on-going project with new material being added gradually.

We have been running an applied mathematics program based on *AM I-III* from first year for the students of chemical engineering at Chalmers since the Fall 99, and we have used parts of the material from *AM IV-* in advanced undergraduate/beginning graduate courses.

Main Features of the Program:

- The program is based on a synthesis of mathematics, computation and application.
- The program is based on new literature, giving a new unified presentation from the start based on constructive mathematical methods including a computational methodology for differential equations.
- The program contains, as an integrated part, software at different levels of complexity.
- The student acquires solid skills of implementing computational methods and developing applications and software using Matlab.
- The synthesis of mathematics and computation opens mathematics education to applications, and gives a basis for the effective use of modern mathematical methods in mechanics, physics, chemistry and applied subjects.
- The synthesis building on constructive mathematics gives a synergetic effect allowing the study of complex systems already in the basic education, including the basic models of mechanical systems, heat conduction, wave propagation, elasticity, fluid flow, electro-magnetism, reaction-diffusion, molecular dynamics, as well as corresponding multi-physics problems.
- The program increases the motivation of the student by applying mathematical methods to interesting and important concrete problems already from the start.
- Emphasis may be put on problem solving, project work and presentation.
- The program gives theoretical and computational tools and builds confidence.
- The program contains most of the traditional material from basic courses in analysis and linear algebra
- The program includes much material often left out in traditional programs such as constructive proofs of all the basic theorems in analysis and linear algebra and advanced topics such as nonlinear systems of algebraic/differential equations.
- Emphasis is put on giving the student a solid understanding of basic mathematical concepts such as real numbers, Cauchy sequences, Lipschitz continuity, and constructive tools for solving algebraic/differential equations, together with an ability to utilize these tools in advanced applications such as molecular dynamics.

- The program may be run at different levels of ambition concerning both mathematical analysis and computation, while keeping a common basic core.

AM I–III in Brief

Roughly speaking, *AM I–III* contains a synthesis of calculus and linear algebra including computational methods and a variety of applications. Emphasis is put on constructive/computational methods with the double aim of making the mathematics both understandable and useful. Our ambition is to introduce the student early (from the perspective of traditional education) to both advanced mathematical concepts (such as Lipschitz continuity, Cauchy sequence, contraction mapping, initial-value problem for systems of differential equations) and advanced applications such as Lagrangian mechanics, n -body systems, population models, elasticity and electrical circuits, with an approach based on constructive/computational methods.

Thus the idea is that making the student comfortable with both advanced mathematical concepts and modern computational techniques, will open a wealth of possibilities of applying mathematics to problems of real interest. This is in contrast to traditional education where the emphasis is usually put on a set of analytical techniques within a conceptual framework of more limited scope. For example: we already lead the student in the second quarter to write (in Matlab) his/her own solver for general systems of ordinary differential equations based on mathematically sound principles (high conceptual and computational level), while traditional education at the same time often focuses on training the student to master a bag of tricks for symbolic integration. We also teach the student some tricks to that purpose, but our overall goal is different.

Constructive Mathematics: Body & Soul

In our work we have been led to the conviction that the constructive aspects of calculus and linear algebra need to be strengthened. Of course, constructive and computational mathematics are closely related and the development of the computer has boosted computational mathematics in recent years. Mathematical modeling has two basic dual aspects: one symbolic and the other constructive-numerical, which reflect the duality between the infinite and the finite, or the continuous and the discrete. The two aspects have been closely intertwined throughout the development of modern science from the development of calculus in the work of Euler, Laplace and Gauss into the work of von Neumann in our time. For

example, Laplace's monumental *Mécanique Céleste* in five volumes presents a symbolic calculus for a mathematical model of gravitation taking the form of Laplace's equation, together with massive numerical computations giving concrete information concerning the motion of the planets in our solar system.

However, beginning with the search for rigor in the foundations of calculus in the 19th century, a split between the symbolic and constructive aspects gradually developed. The split accelerated with the invention of the electronic computer in the 1940s, after which the constructive aspects were pursued in the new fields of numerical analysis and computing sciences, primarily developed outside departments of mathematics. The unfortunate result today is that symbolic mathematics and constructive-numerical mathematics by and large are separate disciplines and are rarely taught together. Typically, a student first meets calculus restricted to its symbolic form and then *much later*, in a different context, is confronted with the computational side. This state of affairs lacks a sound scientific motivation and causes severe difficulties in courses in physics, mechanics and applied sciences which build on mathematical modeling.

New possibilities are opened by creating from the start a synthesis of constructive and symbolic mathematics representing a synthesis of Body & Soul: with computational techniques available the students may become familiar with nonlinear systems of differential equations already in early calculus, with a wealth of applications. Another consequence is that the basics of calculus, including concepts like real number, Cauchy sequence, convergence, fixed point iteration, contraction mapping, is lifted out of the wardrobe of mathematical obscurities into the real world with direct practical importance. In one shot one can make mathematics education both deeper and broader and lift it to a higher level. This idea underlies the present book, which thus in the setting of a standard engineering program, contains all the basic theorems of calculus including the proofs normally taught only in special honors courses, together with advanced applications such as systems of nonlinear differential equations. We have found that this seemingly impossible program indeed works surprisingly well. Admittedly, this is hard to believe without making real life experiments. We hope the reader will feel encouraged to do so.

Lipschitz Continuity and Cauchy Sequences

The usual definition of the basic concepts of *continuity* and *derivative*, which is presented in most Calculus text books today, build on the concept of *limit*: a real valued function $f(x)$ of a real variable x is said to be continuous at $\bar{x}$ if $\lim_{x \rightarrow \bar{x}} f(x) = f(\bar{x})$, and $f(x)$ is said to be differentiable at

$\bar{x}$ with derivative $f'(\bar{x})$ if

$$\lim_{x \rightarrow \bar{x}} \frac{f(x) - f(\bar{x})}{x - \bar{x}}$$

exists and equals $f'(\bar{x})$. We use different definitions, where the concept of limit does not intervene: we say that a real-valued function $f(x)$ is Lipschitz continuous with Lipschitz constant L_f on an interval $[a, b]$ if for all $x, \bar{x} \in [a, b]$, we have

$$|f(x) - f(\bar{x})| \leq L_f |x - \bar{x}|.$$

Further, we say that $f(x)$ is differentiable at $\bar{x}$ with derivative $f'(\bar{x})$ if there is a constant $K_f(\bar{x})$ such that for all x close to $\bar{x}$

$$|f(x) - f(\bar{x}) - f'(\bar{x})(x - \bar{x})| \leq K_f(\bar{x})|x - \bar{x}|^2.$$

This means that we put somewhat more stringent requirements on the concepts of continuity and differentiability than is done in the usual definitions; more precisely, we impose *quantitative* measures in the form of the constants L_f and $K_f(\bar{x})$, whereas the usual definitions using limits are *purely qualitative*.

Using these more stringent definitions we avoid pathological situations, which can only be confusing to the student (in particular in the beginning) and, as indicated, we avoid using the (difficult) concept of limit in a setting where in fact no limit processes are really taking place. Thus, we do not lead the student to definitions of continuity and differentiability suggesting that all the time the variable x is tending to some value $\bar{x}$, that is, all the time some kind of (strange?) limit process is taking place. In fact, continuity expresses that the difference $f(x) - f(\bar{x})$ is small if $x - \bar{x}$ is small, and differentiability expresses that $f(x)$ locally is close to a linear function, and to express these facts we do not have to invoke any limit processes.

These are examples of our leading philosophy of giving Calculus a *quantitative form*, instead of the usual purely qualitative form, which we believe helps both understanding and precision. We believe the price to pay for these advantages is usually well worth paying, and the loss in generality are only some pathological cases of little interest. We can in a natural way relax our definitions, for example to Hölder continuity, while still keeping the quantitative aspect, and thereby increase the pathology of the exceptional cases.

The usual definitions of continuity and differentiability strive for maximal generality, typically considered to be a virtue by a pure mathematician, which however has pathological side effects. With a constructive point of view the interesting world is the constructible world and maximality is not an important issue in itself.

Of course, we do not stay away from limit processes, but we concentrate on issues where the concept of limit really is central, most notably in

defining the concept of a *real number* as the limit of a Cauchy sequence of rational numbers, and a solution of an algebraic or differential equation as the limit of a Cauchy sequence of approximate solutions. Thus, we give the concept of *Cauchy sequence* a central role, while maintaining a constructive approach seeking constructive processes for generating Cauchy sequences.

In standard Calculus texts, the concepts of Cauchy sequence and Lipschitz continuity are not used, believing them to be too difficult to be presented to freshmen, while the concept of real number is left undefined (seemingly believing that a freshman is so familiar with this concept from early life that no further discussion is needed). In contrast, in our constructive approach these concepts play a central role already from start, and in particular we give a good deal of attention to the fundamental aspect of the constructibility of real numbers (viewed as possibly never-ending decimal expansions).

We emphasize that taking a constructive approach does not make mathematical life more difficult in any important way, as is often claimed by the ruling mathematical school of formalists/logicists: All theorems of interest in Calculus and Linear Algebra survive, with possibly some small unessential modifications to keep the quantitative aspect and make the proofs more precise. As a result we are able to present basic theorems such as Contraction Mapping Principle, Implicit Function theorem, Inverse Function theorem, Convergence of Newton's Method, in a setting of several variables with complete proofs as a part of our basic Calculus, while these results in the standard curriculum are considered to be much too difficult for this level.

Proofs and Theorems

Most mathematics books including Calculus texts follow a theorem-proof style, where first a theorem is presented and then a corresponding proof is given. This is seldom appreciated very much by the students, who often have difficulties with the role and nature of the proof concept.

We usually turn this around and first present a line of thought leading to some result, and then we state a *corresponding theorem* as a summary of the hypothesis and the main result obtained. We thus rather use a proof-theorem format. We believe this is in fact often more natural than the theorem-proof style, since by first presenting the line of thought the different ingredients, like hypotheses, may be introduced in a logical order. The proof will then be just like any other line of thought, where one successively derives consequences from some starting point using different hypothesis as one goes along. We hope this will help to eliminate the often perceived mystery of proofs, simply because the student will not be aware of the fact that a proof is being presented; it will just be a logical line of thought, like

any logical line of thought in everyday life. Only when the line of thought is finished, one may go back and call it a proof, and in a theorem collect the main result arrived at, including the required hypotheses. As a consequence, in the Latex version of the book we do use a theorem-environment, but not any proof-environment; the proof is just a logical line of thought preceding a theorem collecting the hypothesis and the main result.

The Mathematics Laboratory

We have developed various pieces of software to support our program into what we refer to as the *Mathematics Laboratory*. Some of the software serves the purpose of illustrating mathematical concepts such as roots of equations, Lipschitz continuity, fixed point iteration, differentiability, the definition of the integral and basic calculus for functions of several variables; other pieces are supposed to be used as models for the students own computer realizations; finally some pieces are aimed at applications such as solvers for differential equations. New pieces are being added continuously. Our ambition is to also add different multi-media realizations of various parts of the material.

In our program the students get a training from start in using Matlab as a tool for computation. The development of the constructive mathematical aspects of the basic topics of real numbers, functions, equations, derivatives and integrals, goes hand in hand with experience of solving equations with fixed point iteration or Newton's method, quadrature, and numerical methods or differential equations. The students see from their own experience that abstract symbolic concepts have roots deep down into constructive computation, which also gives a direct coupling to applications and physical reality.

Go to <http://www.phi.chalmers.se/body soul/>

The *Applied Mathematics: Body & Soul* project has a web site containing additional instructional material and the *Mathematics Laboratory*. We hope that the web site for the student will be a good friend helping to (independently) digest and progress through the material, and that for the teacher it may offer inspiration. We also hope the web site may serve as a forum for exchange of ideas and experience related the project, and we therefore invite both students and teachers to submit material.

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The source of mathematicians pictures is the MacTutor History of Mathematics archive, and some images are copied from old volumes of *Deadalus*, the yearly report from The Swedish Museum of Technology.

My heart is sad and lonely
for you I sigh, dear, only
Why haven't you seen it
I'm all for you body and soul
(Green, Body and Soul)

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