SYSTEMS & SIGNALS

N. LEVAN

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PREFACE

This book has been designed to serve as a text for a Junior-level course in Engineering. It has been used in the Undergraduate Engineering program at UCLA for over six years and has gone through many revisions.

The prerequisites are two years of Calculus, including Differential Equations, and two years of Physics, including Electricity.

There are five chapters which can be covered at a reasonably comfortable pace in one quarter (10 weeks, or approximately 40 contact hours). Chapter 1 begins with the fundamental notion of a System and its Input-Output description, and proceeds quickly to the main properties of the Input-Output transformation. Chapter 2 features the Impulse Response function and its role in the time-domain analysis of linear systems. The Laplace Transform is introduced in Chapter 3, as a tool in the s-domain analysis of linear dynamic systems.

We turn next to Signals, beginning with Fourier Series analysis in Chapter 4. One noteworthy item here is that Mean Square approximation is introduced via the Orthogonality Principle. More advanced topics, such as Fourier Transforms and the Sampling Principle for Band-limited Signals are covered in Chapter 5.

An abundant supply of problems is provided. Each chapter concludes with a set of problems covering the chapter-material. In addition a hundred review problems are listed in the Appendix.

Nhan Levan
Los Angeles
March, 1983

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CHAPTER 1. SYSTEMS: THE INPUT-OUTPUT DESCRIPTION

This chapter is devoted to basic properties of systems with an input-output description. Inputs and outputs -- or signals -- are always taken to be deterministic functions of time.

SYSTEMS: MODEL AND MATHEMATICAL MODEL

The term system is often used loosely. Although a precise definition would be cumbersome, for our purposes we may regard a system as characterized by "inputs" ("stimuli") and "outputs" ("responses").

To study a system one generally begins by forming a model for it. A model is an idealized version of the system and, of course, is not necessarily unique. In other words, a system can admit more than one model, depending on the uses envisaged. Here we are interested only in models described in mathematical terms. Such a description is often called a mathematical model of the system.

For our purposes, a system is represented by a closed box with a number of accessible terminals as depicted in Figure 1.1. Terminals are divided into two groups: input

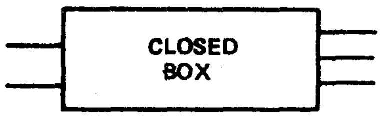

Figure 1.1.

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terminals and output terminals. At input terminals, inputs are applied to the system, while outputs are observed (or measured) at output terminals.

Inputs and outputs -- or signals -- are taken to be time functions, i.e., numerical functions of the time variable t, and are written as $x(\cdot)$ and $y(\cdot)$, respectively. Moreover, they are also deterministic, meaning their values $x(t)$ and $y(t)$ are completely specified for each value of the time variable t.

Let $x(\cdot)$ and $y(\cdot)$ be inputs and outputs of a system, then the pair of functions $(x(\cdot),y(\cdot))$ is called an "input-output" pair. If X and Y are allowable input and output families, then the system is completely characterized by its input-output data, i.e., the family $\{(x(\cdot), y(\cdot)), x(\cdot) \text{ in } X \text{ and } y(\cdot) \text{ in } Y\}$. This is the most general description of a system. Note that, in general, the "output" need not be uniquely determined by the "input"; the same input $x(\cdot)$ may have several outputs $y(\cdot)$ in the family $\{x(\cdot),y(\cdot)\}$.

A system is said to have an input-output description if its output can be expressed completely in terms of the corresponding input function, i.e., for each $x(\cdot)$ there is only one $y(\cdot)$. Such a description conveys the idea that the output is "caused" by the input: the system transforms each input into a unique corresponding output. Thus for a system with an input-output description we can represent its "action" by an (abstract) transformation $T[\cdot]$

acting on an input $x(\cdot)$ to give a unique output $y(\cdot)$:

$$y(\cdot) = T[x(\cdot)], \quad x(\cdot) \text{ in } X, \quad y(\cdot) \text{ in } Y. \tag{1-1}$$

$T[\cdot]$ is called an input-output transformation.

<u>Example</u>

Consider the circuit shown in Figure 1.2. Let $x(\cdot)$ and $y(\cdot)$ be the input voltage and output voltage, respectively. Then a mathematical model for this system is the differential equation:

$$RC\,\frac{dy(t)}{dt} + y(t) = x(t), \qquad t_0 < t < \infty.$$

where the input voltage is applied at some "initial" time t_0 (say).

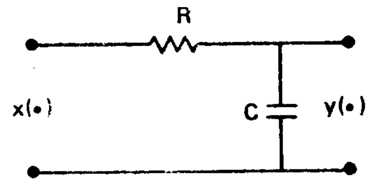

Figure 1.2.

To determine the input-output transformation $T[\cdot]$ in this case, we have to solve the differential equation for $y(\cdot)$. Multiplying both sides of the equation by $e^{\alpha t}$, $\alpha = \frac{1}{RC}$, we find

$$\frac{d}{dt}[e^{\alpha t}\,y(t)] = \alpha e^{\alpha t}\,x(t), \qquad t \geq t_0 .$$

Therefore

$$y(t) = K e^{-\alpha t} + \int_{t_0}^{t} \alpha e^{-\alpha(t-\sigma)} x(\sigma)\, d\sigma , \quad t \geq t_0 ,$$

where K is a constant to be determined. Setting $t = t_0$ we get

$$y(t_0) = K e^{-\alpha t_0} .$$

...e, $K = e^{\alpha t_0} y(t_0)$. Thus the out...

$$) = e^{-\alpha(t-t_0)} y(t_0) + \int_{t_0}^{t} \alpha e^{-\alpha} \ldots$$

$$t \geq t_0$$

It is evident from this relation that the output voltage $y(\cdot)$ depends on the input voltage $x(\cdot)$ and the "initial" condition $y(t_0)$ -- the voltage across the capacitor C at the time t_0. Thus the same input $x(\cdot)$ causes many outputs $y(\cdot)$, depending on different values of $y(t_0)$. Now if we choose $y(t_0) = 0$ then clearly, for $t \geq t_0$:

$$y(t) = \int_{t_0}^{t} \alpha e^{-\alpha(t-\sigma)} x(\sigma)\, d\sigma .$$

We can now say that the circuit transforms each input voltage $x(\cdot)$ into a unique corresponding output voltage $y(\cdot) = T[x(\cdot)]$ -- provided there is no voltage (or, equivalently, no charge) across the capacitor at the time the input voltage was applied, and we write

$$y(\cdot) = T[x(\cdot)] \quad , \quad y(t) = \int_{t_0}^{t} \alpha e^{-\alpha(t-\sigma)} x(\sigma)\, d\sigma \quad ,$$

$$t \geq t_0 .$$

Finally, for the same circuit, if the input is the current $i(\cdot)$, then the mathematical model is simply

$$y(t) = \frac{1}{C} \int_{t_0}^{t} i(\sigma)\, d\sigma + y(t_0) \quad , \quad t \geq t_0 .$$

Therefore, as in the previous case, with $y(t_0) = 0$, [illegible] input-output [illegible]

$$y(\cdot) = T[i(\cdot)] \quad , \quad y(t) = \frac{1}{C} \int_{t_0}^{t} i(\sigma)\, d\sigma \quad , \quad t \geq t_0 .$$

PROPERTIES OF AN INPUT-OUTPUT TRANSFORMATION

Properties of a system are now studied via those of its input-output transformation.

First we define.

Definition

A system with an input-output transformation $y(\cdot) = T[x(\cdot)]$ is said to be linear if

(i) for any scalar k and any $x(\cdot)$ in X:

$$T[kx(\cdot)] = kT[x(\cdot)] \; ;$$

(ii) for any $x_1(\cdot)$ and $x_2(\cdot)$ in X:

$$T[x_1(\cdot) + x_2(\cdot)] = T[x_1(\cdot)] + T[x_2(\cdot)] \; .$$

It is evident that (i) and (ii) can be combined into the single condition

$$T[k_1x_1(\cdot) + k_2x_2(\cdot)] = k_1T[x_1(\cdot)] + k_2T[x_2(\cdot)] \qquad (1.2)$$

for any scalars k_1, k_2 and any $x_1(\cdot)$, $x_2(\cdot)$ in X.

We must note that in the above we have assumed that $kx(\cdot)$ and $x_1(\cdot) + x_2(\cdot)$ are in X for each k and for any $x(\cdot)$, $x_1(\cdot)$ and $x_2(\cdot)$ in X. This is the same as saying that X is a linear space. Similarly Y is also a linear space, and the above Definition means that $T[\cdot]$ is a <u>linear transformation</u> from X to Y.

It follows at once from (1.2) that $T[0] = 0$. Thus a linear system -- in the sense of the above Definition -- must be such that <u>a "zero" input (x=0) always results in a "zero" output (y=0)</u>. This implies that we are only concerned with the class of systems which are at <u>"rest"</u> -- i.e., zero input results in zero output -- at the time an input is applied to the systems.

When a system does not have a linear input-output transformation it is said to be <u>nonlinear</u>.

The next important property is that of a <u>time-invariant (or fixed, or stationary)</u> system. Heuristically speaking, a system is time-invariant if an input is shifted along the time axis by an amount τ (say), then the corresponding output is shifted by the same amount as illustrated in Figure 1.3. Clearly $x(t-\tau)$ for $\tau > 0$ (resp. $\tau < 0$) is just $x(t)$ shifted to the right (resp.

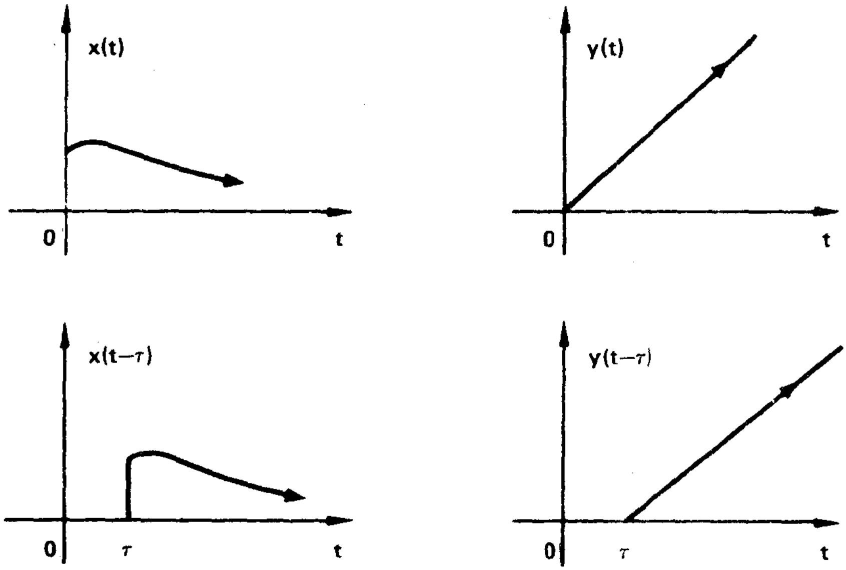

Figure 1.3.

left) by the amount τ. We therefore have the next Definition.

<u>Definition</u>

A system with an input-output transformation $y(\cdot) = T[x(\cdot)]$ is time-invariant if, for any t and any τ:

$$y(t) = T[x(t)]$$

and

$$z(t) = T[x(t-\tau)] \quad .$$

Then

$$z(t) = y(t-\tau) \quad .$$

Example

Consider

$$y(\cdot) = T[x(\cdot)] \quad , \quad y(t) = \int_{-\infty}^{\infty} (t-\sigma)\ x(\sigma)\ d\sigma \quad ,$$

$$-\infty < t < \infty \quad .$$

We have

$$T[x(\cdot-\tau)] = \int_{-\infty}^{\infty} (t-\sigma)\ x(\sigma-\tau)\ d\sigma = z(t)$$

and

$$y(t-\tau) = \int_{-\infty}^{\infty} (t-\tau-\sigma)\ x(\sigma)\ d\sigma \quad .$$

Therefore $z(t) = y(t-\tau)$ and the system is time-invariant.

If a system is not time-invariant then it is called time-varying.

It is important to note that for a time-invariant system shifting an input along the time axis does not change the shape of the corresponding output. Therefore, for time-invariant systems, the origin of time can always be taken to be 0. In other words, if $x(\cdot)$ is applied to a time-invariant system at some time t_0 ($\neq 0$), then we can always shift it to the origin and take 0 to be the time at which the input is applied to the system. Consequently we can take $x(t)$ to be 0 for $t < 0$.

The next important concept is that of causality or physical realizability. A system is said to be causal if the value $y(t)$ at any time t of an output $y(\cdot)$ depends

only on the values of an input $x(\cdot)$ up to the time t -- **i.e., the values** $x(\sigma)$ for each $\sigma \leq t$. Thus if we **regard** t **as the** <u>**present time**</u> -- i.e., "now" -- then for a causal system present value of an output can only depend on past and present values of the input that causes it -- but not on the future values of the input.

If for a causal system and for any t, $y(t)$ only depends on $x(t)$ -- i.e., present value of output only depends on present value of input -- then the system is said to be <u>instantaneous</u> or <u>memoryless</u> (or [illegible]). Otherwise it is said to have memory.

<u>**Example**</u>

The system with the input-output transformation

$$y(\cdot) = T[x(\cdot)] , \quad y(t) = \int_{-\infty}^{t} x(\sigma)\, d\sigma , \quad -\infty < t < \infty$$

is causal and has memory, while

$$y(t) = \int_{0}^{\infty} e^{-(t-\sigma)}\, x(\sigma)\, d\sigma , \qquad -\infty < t < \infty$$

is not causal. The system defined by

$$y(\cdot) = T[x(\cdot)] , \quad y(t) = 4x(t) + 2$$

is memoryless.

Finally, if the time variable t -- of the input and output signals -- takes all values in an interval (finite or nonfinite) then the system is said to be continuous-time. If t takes only discrete values then the system is called discrete-time.

PROBLEMS

1. Verify whether the following input-output transformations are: linear, nonlinear, time-invariant, time-varying, causal, noncausal, or memoryless:

$$y(t) = t x(t), \qquad 0 \le t \le 5;$$

$$y(t) = \int_{-\infty}^{t} e^{-(t-\sigma)} \sigma x(\sigma)\, d\sigma, \qquad -\infty < t < \infty;$$

$$y(t) = x(t) + \int_{0}^{t} (t-\tau)\, x(\tau)\, d\tau, \qquad t \ge 0;$$

$$y(t) = \frac{dx(t)}{dt} - \int_{t}^{\infty} t\, \tau^2\, x(\tau)\, d\tau, \qquad t > 0;$$

$$y(t) = 4x(t)^2,$$

$$y(t) = x(t-5), \qquad -\infty < t < \infty;$$

$$y(t) = \int_{-\infty}^{\infty} t\, \sigma x(\sigma)\, d\sigma, \qquad -\infty < t < \infty.$$