Variational Methods in Mathematics, Science and Engineering

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PREFACE

There were three main impulses which led me to the writing of the present book: My many years of lecturing in special courses for selected students and graduate students-engineers at the College of Civil Engineering of the Technical University in Prague, frequent consultations with technicians and physicists who have asked for advice in overcoming difficulties encountered in solving their problems, and finally many discussions with mathematicians themselves, "pure" as well as "applied". This is also why the book is proposed for a relatively wide range of readers.

The first half of the book (Parts I–III) is devoted to the theory based on the theorem on the minimum of the functional of energy and contains current variational methods with examples of their applications. These problems are relatively familiar to engineers and physicists. The second half (Part IV–VI) is grounded on the rather more abstract Lax-Milgram theorem. Roughly speaking, it is written "more" for mathematicians while the first half is written "more" for consumers of mathematics. Nevertheless, I tried very much that the whole book be for use and interest to both categories of readers.

The realization of such a conception is not at all simple. Indeed, the mentioned categories of readers have often quite opposing ideas as concerns such a book and, consequently, different requirements which cannot be met simultaneously. For example, mathematicians cannot be met in their desire that the book be written in concise mathematical form, with sufficiently rapid gradation. In fact, for a reader who is not a professional mathematician and who has to understand well all the individual relations and consequences (or who even intends to engage himself in the creative development of the theory), it is necessary to become acquainted with modern mathematical tools, at least with the foundations of functional analysis. This mathematical discipline – familiar to a mathematician – brings a lot of abstract concepts to a non-mathematician which cannot be absorbed with haste. This is why I have advanced very cautiously. To begin with, I have defined the inner product, the norm and the metrics on the set of sufficiently smooth functions where these concepts – discussed later in more general functional spaces – are rather intuitive. Further concepts were prepared in the space L_2, one of the most simple Hilbert functional spaces. Only then did I proceed to the definition of the abstract Hilbert space. This inductive rather than deductive approach has been applied in many other places of the book, even in its (more abstract) second half.

As concerns mathematicians, I have applied my best efforts to prepare topics of interest also for them. In the book they find sufficiently general existence theorems on the one hand and a lot of my own results, some of them published here for the first time (e.g., the new methods discussed in Chaps. 43 and 45) on the other hand. Some of my results have been published only here. To mention some of them: refined estimates in inequalities of the Friedrichs type (Chap. 18), new results in eigenvalue problems and eigenvalue estimates for elliptic equations of the form $Au - \lambda Bu = 0$ (Chaps. 39, 40), etc. See also the untraditional approach to the problems of Chaps. 19, 34, 35, 44 etc.

Considerable attention has been devoted to practical aspects of the methods (see the complete numerical solution of examples in Chaps. 21, 26 and 41) as well as to the questions of numerical stability and of the proper choice of a base. In this connection, I have pointed out the practical advantages and disadvantages of the individual methods, including the finite element methods.

The question of the error estimate of an approximate solution represents a rather difficult problem. One of the possibilities of establishing this estimate is given by the method of orthogonal projections or the Trefftz method (Chap. 44). However, a number of drawbacks of practical nature are encountered here. Therefore, I have tried to develop a rather simple estimate (11.21) and to increase its efficiency namely by finding more effective estimates for the constant of positive definiteness C (Chap. 18).

As mentioned above, the second half of the book and especially Part IV is to a certain degree more abstract. Especially, I have sought to analyse in detail problems (nonsymmetric in general) in differential equations with nonhomogenous boundary conditions, including the Neumann problem for higher order equations. This is a complex of problems which happens to be rather cumbersome in the theory based on the theorem on the minimum of functional of energy. At the same time, I have tried to investigate these topics without introducing the so-called factor spaces, which are conceptually difficult for a technically oriented reader.

By mentioning Part IV as relatively more difficult I do not mean to say that it is dedicated to mathematicians only. On the contrary, my aim was to present the problems there in sufficient detail so that it be well comprehensible even to technicians and physicists to whom it may bring – in my opinion – precisely that what they find not in the "classical" theory of variational methods.

Part V is devoted to eigenvalue problems for elliptic equations of the form $Au - \lambda u = 0$, or, more generally, of the form $Au - \lambda Bu = 0$.

In Part VI several special methods are presented. It was my intention to describe these methods in such a way that the reader interested in their application be able to understand them without necessarily reading the more complicated Part IV.

In the final Chapter 47, I have attempted to provide the reader with information concerning several other topics related to the discussed complex of problems (nonlinear problems, problems on infinite domains, etc.).

For the convenience of the reader, a table of functionals is added at the end of the book for the most common types of problems from the theory of differential equations with boundary conditions, including the corresponding Ritz system of equations.

Finally, a brief note concerning terminology. There is a significant disunity in using concepts such as linear set, linear space, etc., in the literature. Some of the authors understand by "linear space" a set of elements possessing certain properties of linearity, while others denote by the same terms the set in which there is already introduced a certain (linear) metric or topology. In this publication we understand by a linear space a linear metric space. By a subspace of a complete linear metric space a complete linear subspace is always understood.

In concluding the preface, I sincerely wish to thank all those who have by their work or advice contributed to the improvement of this book.

Prague, 1979 Karel Rektorys

In Part VI several special methods are presented, some of them entirely new. It was my intention to describe these methods in such a way that the reader interested in their application would be able to understand them without the necessity of reading the more complicated Part IV in advance.

By mentioning Part IV as relatively more difficult I do not mean to say that it is dedicated to mathematicians only. On the contrary, my aim was to present the problems there in sufficient detail so that it be well comprehensible even to technicians and physicists to whom it may bring – in my opinion – precisely that what they find not in the "classical" theory of variational methods.

In the concluding Chap. 47, I have attempted to provide the reader with information concerning several other topics related to the discussed complex of problems (nonlinear problems, problems on infinite domains, etc.).

For the convenience of the reader, a table of functionals is added at the end of the book for the most common types of problems from the theory of differential equations with boundary conditions, including the corresponding Ritz system of equations.

Let me add a short comment concerning terminology: There seems to be a significant disunity when using concepts such as linear set, linear space, etc., in the Czech as well as in the foreign literature. Some of the authors understand by "linear space" a set of elements possessing certain properties of linearity, while others denote by the same term the set in which there is already introduced a certain (linear) metric or topology. In this publication we understand by a linear space a linear *metric* space. By a subspace of a complete linear metric space a *complete* linear subspace is always understood.

In concluding the preface, I sincerely wish to thank all those who have by their work or advice contributed to the improvement of this book.

Prague, January 27th, 1972.

Karel Rektorys

NOTATION FREQUENTLY USED

$u \in M$ (or $u \notin M$)	u belongs (or does not belong) to the set M.
E_N	N-dimensional Euclidean space with the usual definition of the distance $\varrho(A, B)$ of two points $A(a_1, \ldots, a_N)$, $B(b_1, \ldots, b_N)$, $\varrho(A, B) = \sqrt{[(b_1 - a_1)^2 + \ldots + (b_N - a_N)^2]}$.
G	domain in E_N, i.e., an open connected set in E_N. In this book, we consider bounded domains only with the so-called Lipschitz boundary (Chap. 2, p. 21; Chap. 28, p. 324). In the case of $N = 1$, the domain G is an open interval (a, b).
Γ	boundary of the domain G.
$\bar{G}$	closure of the set G in the space E_N (thus, $\bar{G} = G + \Gamma$). Instead of the closure of the domain G in the space E_N we speak, in brief, of the closed domain $\bar{G}$.
$u(x)$	brief notation for $u(x_1, \ldots, x_N)$.
$\int_G u(x)\,dx$	brief notation for $\int \ldots \int_G u(x_1, \ldots, x_N)\,dx_1 \ldots dx_N$. If $N = 1$, then $\int_G u(x)\,dx = \int_a^b u(x)\,dx$.
$C^{(k)}(G)$	the set of functions $u(x)$ whose (partial) derivatives up to the k-th order inclusive are continuous in G. By the derivative of order zero we understand the function $u(x)$. Instead of $C^0(G)$ we write $C(G)$. Thus, $u \in C(G)$ means that the function $u(x)$ is continuous in G; $u \in C^{(1)}(G)$ means that the functions $u(x)$, $\partial u/\partial x_1, \ldots, \partial u/\partial x_N$ are continuous in G.
$C^{(k)}(\bar{G})$	the set of functions $u(x)$ whose (partial) derivatives up to the k-th order inclusive are continuous in $\bar{G}$. Instead of $C^{(0)}(\bar{G})$ we write $C(\bar{G})$.
$C^{(\infty)}(\bar{G})$ [or $\mathscr{E}(\bar{G})$]	the set of functions $u(x)$ whose (partial) derivatives of all orders are continuous in $\bar{G}$.
$C_0^{(\infty)}(G)$ [or $\mathscr{D}(G)$]	the set of functions with compact supports in G. See Chap. 8, p. 99.
H	Hilbert space, Chap. 6, p. 73.
(u, v)	inner (scalar) product of the elements u, v in a Hilbert (or pre-Hilbert) space.
$u \perp v$	the elements u, v of a Hilbert space are orthogonal; $u \perp v \Leftrightarrow (u, v) = 0$.
$H = H_1 \oplus H_2$	decomposition of the Hilbert space H into an orthogonal sum of subspaces H_1, H_2, p. 79.

$H_2 = H \ominus H_1$	orthogonal complement to the subspace H_1 in the Hilbert space H.
$\|u\|$	norm of the element u, pp. 24, 34, 70, 83.
$\varrho(u, v)$	distance of the elements u, v, pp. 27, 34, 70, 81.
$L_2(G)$	Hilbert space of functions square integrable in the domain G, Chap. 3.
$W_2^{(k)}(G)$	Hilbert space whose elements are those functions from $L_2(G)$ which have generalized derivatives in G up to the k-th order inclusive, Chap. 29.
$\mathring{W}_2^{(k)}(G)$	closure of the set $C_0^{(\infty)}(G)$ in the metric of the space $W_2^{(k)}(G)$, Chap. 30.
$H^k(G) = W_2^{(k)}(G)$	p. 335.
$L_2(\Gamma)$	p. 327.
$W_2^{(k)}(\Gamma)$	p. 340.
V	Def. 32.1, 32.2. See also p. 359.
D_A	domain of the operator A.
$(u, v)_A$	inner product defined in Chap. 10, p. 121.
H_A	Hilbert space with the inner product $(u, v)_A$, Chap. 10.
$A(v, u)$	bilinear form corresponding to the operator A, Chap. 32, p. 369.
$((v, u)) = A(v, u) + a(v, u)$	Chap. 32, p. 375.
$\{v; P\}$	set whose elements have the property P. For instance, we read $M = \{v; v \in C^{(2)}(\bar{G}), v = 0 \text{ on } \Gamma\}$ as follows: M is the set of all functions which belong to $C^{(2)}(\bar{G})$ and are equal to zero on the boundary Γ. If the elements v belong to some metric space, it is usual to understand by the above symbol the set M *with the metric of that space* (thus, a metric space as well). E.g. $\left\{v; v \in L_2(G), \int_G v(x)\,dx = 0\right\}$ is the subspace of the space $L_2(G)$ the elements of which are those functions from $L_2(G)$ for which $\int_G v(x)\,dx = 0$ holds.