VECTOR AND TENSOR ANALYSIS

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PREFACE

This book is intended for an introductory course in vector and tensor analysis. In writing the book, the author's objective has been to acquaint the students with the various fundamental concepts of vector and tensor analysis together with some of their corresponding physical and geometric interpretations, as well as to enable the students to attain some degree of proficiency in the manipulation and application of the mechanics and techniques of the subject.

Throughout the book, we place great emphasis on intuitive understanding as well as geometric and physical illustrations. To help achieve this end, we have included a great number of examples drawn from the physical sciences, such as mechanics, fluid dynamics, and electromagnetic theory, although prior knowledge of these subjects is not assumed. We stress the development of basic techniques and computational skills and deliberately de-emphasize highly complex proofs. Teaching experience at this level suggests that highly technical proofs of theorems are difficult for students and serve little purpose toward understanding the significance and implications of the theorems. Thus we have presented the classical integral theorems of Green, Gauss, and Stokes only intuitively and in the simplest geometric setting. At the end of practically every section, there are exercises of varying degrees of difficulty to test students' comprehension of the subject matter presented and to make the students proficient in the basic computation and techniques of the subject.

The book contains more than enough material for a one-year or two-quarter course at the junior or senior level or even at the beginning graduate level for physical sciences majors. Omitting Secs. 3.9 through 3.12, Chaps. 1 through 4 can serve as material for a one-semester course in vector analysis, or for a one-quarter course with further deletion of topics depending on the interest of the class. Preceded by Secs. 3.9 and 3.11, the material of Chaps. 5 and 6 can then be used for a second-semester or a one-quarter course in tensor analysis.

As a prerequisite for a course based on this book, the students must be familiar with the usual topics covered in a traditional elementary calculus course. Specifically, the students must know the basic rules of differentiation and integration, such as the chain rule, integration by parts, and iterated integration of multiple integrals. Although a knowledge of matrix algebra would be helpful, this is not an essential prerequisite. The book requires only the bare rudiments of this subject, and they are summarized in the text.

The author wishes to thank his colleagues Professor Steven L. Blumsack, Wolfgang Heil, David L. Lovelady, and Kenneth P. Yanosko for reviewing portions of the manuscript and offering valuable comments and suggestions, and Professors Chiu Yeung Chan and Christopher K. W. Tam for testing the material on tensors in their classes during the developmental stage of the book. Last but not least, the author acknowledges with gratitude the assistance rendered by the production and editorial department of the publisher.

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CONTENTS

Chapter 1

VECTOR ALGEBRA

1.1 INTRODUCTION

In the study of physics, we encounter quantities such as volume and temperature, which can be described by the specification of their magnitude alone in terms of some appropriate units. For example, the volume of a cube can be described by the number of cubic inches, and the temperature at a particular time of a day can be described by giving the number of degrees on a Fahrenheit scale. Such quantities characterized by the fact that they have magnitude only are called scalar quantities, and they are represented by real numbers (also called scalars). On the other hand, there are other physical quantities such as displacement, force, velocity, and acceleration which cannot be described by single numbers. These quantities possess not only magnitude but also direction so that a complete description of any such quantity must specify these two pieces of information. Thus when a weatherman reports the wind velocity on a particular day, he specifies not only the speed of the wind (magnitude of the wind velocity) but also the direction in which the wind is blowing. Such quantities characterized by having magnitude and direction are called vector quantities.

Just as we use real numbers or scalars to represent and manipulate scalar quantities, so we use the mathematical entities called vectors to represent and manipulate vector quantities. Thus, in a sense, vectors can be thought of as generalized numbers. The study of the representation of vectors, the algebra and calculus of vectors, and their various applications constitute the subject matter of vector analysis.

Scalars and vectors are hardly sufficient to treat the class of quantities that are of interest in applied mathematics and physics. In fact, there are quantities of a more complicated structure whose description requires more than knowledge of a magnitude and a direction. For example, to describe a quantity such as stress, we need to give a force and a surface on which the force acts. Such a quantity can be described and represented only by the mathematical entity called tensor. As we see later, vectors and scalars are actually special cases of tensors.

In this book, we study vectors and tensors in the familiar three-dimensional Euclidean space. In many cases, the concepts and results obtained for the three-dimensional space can be immediately extended to higher dimensional spaces. Throughout the book we use underscored letters--$\underline{A},\underline{B},\ldots$ or $\underline{a},\underline{b},\ldots$--to denote vectors and lowercase letters a,b,... to denote real numbers or scalars. Tensors are represented by their so-called components.

1.2 DEFINITION OF A VECTOR

A vector may be defined in essentially three different ways: geometrically, analytically, and axiomatically. The geometric definition makes use of the notion of a directed line segment or an arrow. A line segment determined by two given points P and Q is said to be directed if one of the points, say P, is designated as the initial point and the other, Q, the terminal point. The directed line segment so obtained is then denoted by PQ, and it is shown graphically by drawing an arrow from P to Q (Fig. 1.1). The length of PQ is denoted by $|PQ|$. If PQ and RS are two directed line segments, then they are said to be equal (PQ = RS), if they have the same length and the same direction. Now, geometrically, a vector is defined as the collection of all directed line segments or arrows having the same length and direction. (Such a collection is also called an equivalence class of directed line segments, and any

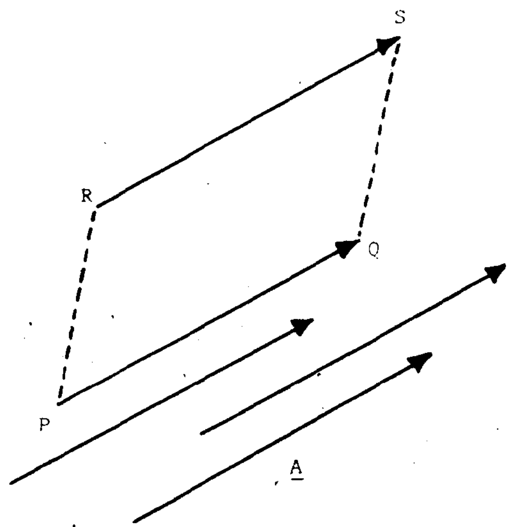


FIG. 1.1 Directed line segments.

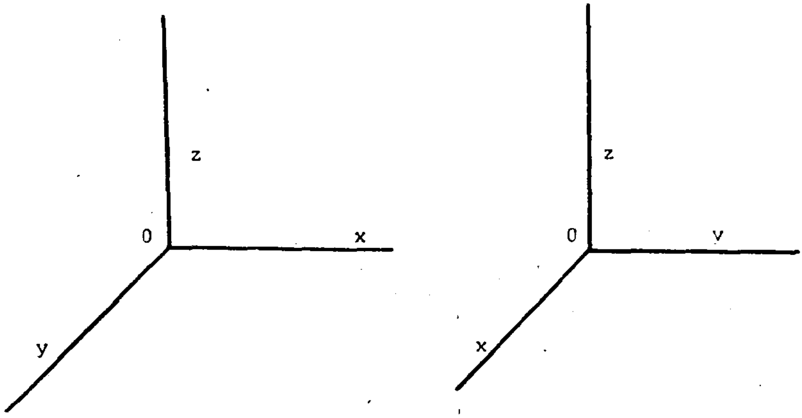
two members of the class are said to be equivalent.) The common length of the arrows represents the magnitude of the vector, and the arrowhead indicates the direction of the vector. Any one of the arrows in the collection can be identified as the vector. Thus, for example, the collection of arrows as shown in Figure 1.1 defines a vector $\underline{A}$. In this definition, the algebraic operations on vectors are introduced and studied geometrically. This approach has the advantage of being free of any frame of reference or coordinate system, which is an important property of vectors. However, it is inefficient for computational purposes.

In the analytic approach, a vector is defined as an ordered triple of real numbers $[a_1, a_2, a_3]$ relative to a given coordinate system. The real numbers a_1, a_2, a_3 are called the components of the vector. These components, of course, arise naturally from the geometric description of a vector once a coordinate system is introduced. Algebraic operations on vectors are then defined in terms of the components, and the properties of these operations are readily deduced from the corresponding properties of the real numbers.

By far, this approach is the most convenient for theoretical and practical considerations. However, it must be kept in mind that the components are dependent on the coordinate system used, and, therefore, a change of coordinate system results in a change of the components, although the vector itself remains the same.

Lastly, the axiomatic point of view treats a vector simply as an undefined entity of an abstract algebraic system called a linear vector space. In such a system, the vectors are required to satisfy certain sets of axioms with respect to two algebraic operations that are undefined concepts. As we see later, the sets of axioms for a linear vector space are precisely the properties satisfied by vectors with respect to the vector operations of addition and multiplication by scalars as developed by either the geometric or the analytic approach.

In this chapter, we develop the algebra of vectors on the basis of the analytic definition of a vector, and we use directed line segments or arrows to represent vectors geometrically and to give geometric interpretations of our results. Accordingly, we assume a coordinate system in our space. As it is commonly used, we assume a right-handed rectangular cartesian coordinate system (x,y,z). The student may recall that such a coordinate system consists of three lines that are perpendicular to each other at a common point 0 called the origin (Fig. 1.2). The lines are designated as the x, y, and z coordinate axes, and a definite direction on each axis is chosen as the positive direction. The coordinate system is then said to be right-handed if when the index finger of the right hand points along the positive x-axis and the center finger points along the positive y-axis, the thumb points along the positive z-axis. This rule is known as the right-hand rule. A different version of this rule states that if the fingers of the right hand point in the direction in which the positive x-axis must be rotated (through the smaller angle $\theta = 90°$) in order to coincide with the positive y-axis, the thumb points in the direction of the positive z-axis. With respect to such a rectangular cartesian coordinate system, we now define a vector as follows:



(a) Left-handed (b) Right-handed

FIG. 1.2 Rectangular cartesian coordinate systems.

DEFINITION 1. A vector $\mathbf{A}$ is an ordered triple of real numbers a_1, a_2, a_3, written as $\mathbf{A} = [a_1,a_2,a_3]$; a_1 is called the first or x-component, a_2 the second or y-component, and a_3 the third or z-component of the vector.

A vector whose components are all zero is called the zero vector, and it is denoted by $\mathbf{0}$; thus, $\mathbf{0} = [0,0,0]$. The negative of a vector $\mathbf{A}$, denoted by $-\mathbf{A}$, is defined as the vector

$$-\mathbf{A} = [-a_1,-a_2,-a_3]$$

For example, if $\mathbf{A} = [2,-1,3]$, then $-\mathbf{A} = [-2,1,-3]$.

DEFINITION 2. The magnitude of a vector $\mathbf{A} = [a_1,a_2,a_3]$, denoted by $|\mathbf{A}|$, is the real number defined by

$$|\mathbf{A}| = (a_1{}^2 + a_2{}^2 + a_3{}^2)^{1/2} \qquad (1.1)$$

It is clear that $|\mathbf{A}| \geq 0$ and that $|\mathbf{A}| = 0$ if and only if $\mathbf{A} = \mathbf{0}$. A vector that has a magnitude equal to 1 is called a unit vector.

In the sequel, unless stated otherwise, every vector under discussion is assumed to be nonzero.

1.3 GEOMETRIC REPRESENTATION OF A VECTOR

A vector (nonzero) $\underline{A} = [a_1, a_2, a_3]$ can be represented geometrically by a directed line segment or an arrow drawn from the origin to the point $P:(a_1, a_2, a_3)$ as shown in Figure 1.3. In fact, we see that the length of the arrow is equal to $(a_1^2 + a_2^2 + a_3^2)^{1/2}$, which is the magnitude of the vector as defined in (1.1). The direction of OP may be described by the three numbers cos α, cos β, cos γ that are called the direction cosines of OP. From Figure 1.3, we see that

$$\cos \alpha = \frac{a_1}{|\underline{A}|}, \qquad \cos \beta = \frac{a_2}{|\underline{A}|}, \qquad \cos \gamma = \frac{a_3}{|\underline{A}|} \tag{1.2}$$

so that the direction cosines of OP are proportional to the components of the vector. Thus the directed line segment OP represents the vector $\underline{A}$ in magnitude as well as in direction. We call OP a geometric vector representing $\underline{A}$.

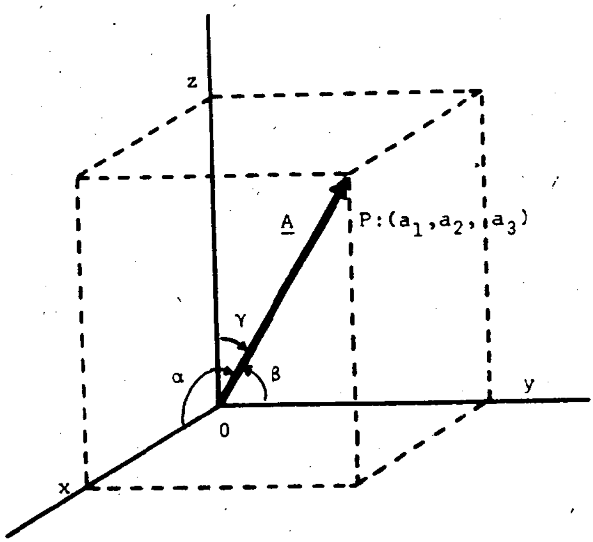


FIG. 1.3 Geometric representation of a vector.

Notice that by (1.1) the direction cosines (1.2) satisfy the important relation

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \tag{1.3}$$

Thus the vector $\underline{u} = [a_1/|\underline{A}|,\ a_2/|\underline{A}|,\ a_3/|\underline{A}|]$ is a unit vector having the same direction as $\underline{A}$. It follows that every nonzero vector can be converted into a unit vector (normalized) by dividing its components by its magnitude. This process of making an arbitrary vector into a unit vector is sometimes called the normalization process.

It should be pointed out that a vector can also be represented by an arrow drawn from an arbitrary point in space. In fact, if P is a point with coordinates (x_0, y_0, z_0) and Q another point with the coordinates $(x_0 + a_1,\ y_0 + a_2,\ z_0 + a_3)$, then the directed line segment PQ also represents the vector $\underline{A}$ geometrically (Fig. 1.4). This can be readily checked in the same manner discussed previously.

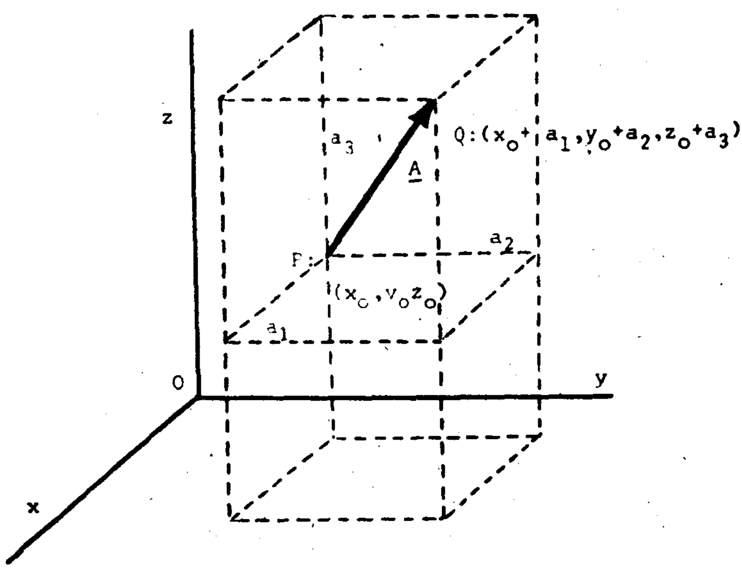


FIG. 1.4 Representation of a vector.

In this case, we note that the components of $\underline{A}$ are given by the differences between the corresponding coordinates of the points P and Q. Thus a vector represented by a directed line segment with initial point (x_1,y_1,z_1) and terminal point (x_2,y_2,z_2) has components given by

$$a_1 = x_2 - x_1, \qquad a_2 = y_2 - y_1, \qquad a_3 = z_2 - z_1 \tag{1.4}$$

Therefore, in representing a vector by a directed line segment, the choice of the initial point is immaterial; what matters is the length and the direction. However, it is convenient to choose the initial point at the origin so that the components of the vector coincide with the coordinates of the terminal point. In this way, we then associate a vector with every point in space in a one-to-one fashion.

We observe that if PQ represents a vector $\underline{A}$, then QP represents the negative of the vector, that is, $-\underline{A}$. Obviously, the zero vector is represented simply by a point, the origin. The zero vector is the only vector that does not have a direction.

In two-dimensional space (a plane), a vector consists of only two components. In other words, a vector in a plane is an ordered pair of real numbers, $\underline{A} = [a_1,a_2]$, with respect to a rectangular cartesian coordinate system (x,y). Such a vector is represented geometrically by an arrow drawn from the origin to the point (a_1,a_2) The direction of the arrow is uniquely determined by the angle $\theta = \arctan(a_2/a_1)$, see Figure 1.5. The relationship between the components of the vector and the length and direction of an arrow representing the vector is given by

$$a_1 = A \cos\theta, \qquad a_2 = A \sin\theta \tag{1.5}$$

where A indicates the length of the arrow.

EXAMPLE 1. A vector $\underline{A}$ is represented by the directed line segment PQ, where P:(2,-1,3) and Q:(-1,-2,4). Find the components and magnitude of the vector.

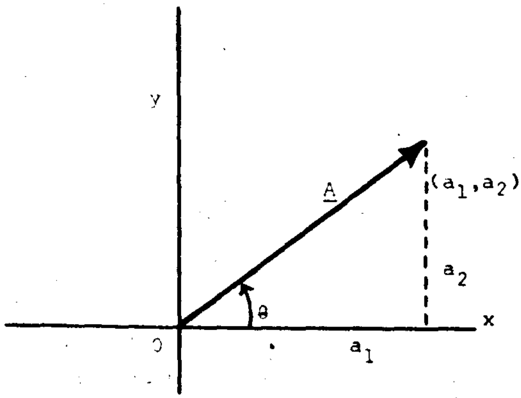


FIG. 1.5 Vector in the plane.

Solution: Let us denote the components of the vector by a_1, a_2, and a_3. Then by (1.4), we have

$$a_1 = -1 - 2 = -3, \qquad a_2 = -2 - (-1) = -1, \qquad a_3 = 4 - 3 = 1$$

Hence, by (1.1), the magnitude of the vector is equal to

$$|\underset{\sim}{A}| = \sqrt{(-3)^2 + (-1)^2 + 1^2} = \sqrt{11}$$

EXAMPLE 2. Find the direction cosines of a directed line segment which represents the vector $\underset{\sim}{A} = [2,-1,2]$.

Solution: The magnitude of the vector is equal to

$$|\underset{\sim}{A}| = \sqrt{2^2 + (-1)^2 + 2^2} = 3$$

Hence, according to (1.2), the direction cosines of a directed line segment representing $\underset{\sim}{A}$ are given by

$$\cos\alpha = \frac{2}{3}, \qquad \cos\beta = -\frac{1}{3}, \qquad \cos\gamma = \frac{2}{3}$$

EXAMPLE 3. If $\underset{\sim}{A} = [-1,\sqrt{3}]$, what is the length and direction of an arrow which represents the vector?

Solution: The length of an arrow representing the vector is equal to the magnitude of the vector, which is given by