

**HANDBOOK OF
BOOLEAN ALGEBRAS**

VOLUME 3

Edited by

J. DONALD MONK

ROBERT BONNET

HANDBOOK OF BOOLEAN ALGEBRAS

VOLUME 3

Edited by

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with the cooperation of

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Université Claude-Bernard, Lyon I

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Section C

SPECIAL CLASSES OF BOOLEAN ALGEBRAS

The four chapters of this Section give an in-depth coverage of some important classes of Boolean algebras discussed in Part I.

Chapter 19, Superatomic Boolean algebras, by Judy Roitman, goes into two facets of the theory of these algebras which have been widely studied: the problem of existence of superatomic algebras with a long cardinal sequence, each term of which is small (thin-tall algebras and related concepts), and the realization of automorphism groups in superatomic algebras in a natural way.

Chapter 20, Projective Boolean algebras, by Sabine Koppelberg, gives some characterizations of these algebras, Shchepin's characterization of when a projective algebra is free, and discusses the number of isomorphism types of them.

Chapter 21, Countable Boolean algebras, by R.S. Pierce, is a comprehensive treatment of this important class of algebras. It deals with several kinds of invariants which have been induced for countable algebras, the structures induced on the isomorphism types of these algebras by the product and coproduct constructions -- culminating in the theorems of Dobbertin, Ketonen, and Trnková -- and a discussion of the notion of primitive Boolean algebra induced by Hanf.

Chapter 22, Measure algebras, by David H. Fremlin, gives many of the known results about this important class of algebras; Maharam's theorem, liftings, a discussion of which BAs can have measures are discussed, and other topics are treated.