

Plasma Spectroscopy

Hans R. Griem

53.732
G545

Plasma Spectroscopy

Hans R. Griem

*Professor of Physics
University of Maryland
College Park, Maryland*

McGraw-Hill Book Company
New York San Francisco Toronto London



Plasma Spectroscopy

**Copyright © 1964 by McGraw-Hill, Inc.
All Rights Reserved. Printed in the
United States of America. This book,
or parts thereof, may not be
reproduced in any form without
permission of the publishers.**

***Library of Congress Catalog Card
Number 63-23250***

24680

56961

Preface

This book is meant as a reference source for spectroscopic research in experimental plasma physics and the physics of stellar atmospheres. It may also serve as a text for a special graduate course in physics or astronomy, as it contains an introduction to the general theory of quantitative spectroscopy, numerical results of this theory and their derivation, and a description of experimental techniques and results. Applications of theoretical and experimental methods to the determination of plasma conditions or atomic parameters are equally stressed. The extensive tabulations of oscillator strengths, continuum emission coefficients, and line broadening parameters should aid in laboratory experiments or quantitative analyses of stellar spectra. For ease of reference, the tables are grouped together after the text. The problems attached to the fifteen chapters are designed as guides to a quantitative understanding of methods and applications; the solutions appear at the end of the book.

Quantitative spectroscopic techniques can be applied to many problems in laboratory plasma physics, astronomy, and related fields. Most of the methods require a knowledge of atomic spectroscopy, which has been treated in a number of books (see the bibliography below). Accordingly, the reader of this book is assumed to have sufficient familiarity with the theory of atomic spectra. Furthermore, the reader should have been exposed to quantum-mechanical perturbation theory and to certain aspects of statistical mechanics.

Although the original literature on plasma spectroscopy has developed to an impressive size, no comprehensive and critical review has been attempted heretofore. For this reason, there will without doubt be omissions and errors in this book, especially in the tabulated material. (The author would be grateful if such defects were pointed out to him.) The reader will also notice that in some areas much remains to be done. Here the hope is that careful measurements and theoretical investigations can be stimulated to remedy this situation in the future.

The author was introduced to this challenging subject by his teachers, W. Lochte-Holtgreven and A. Unsöld, at Kiel University. The present book has grown out of work done by the plasma physics groups at the University of Maryland and the U.S. Naval Research Laboratory for the last six or seven years. Thanks are due many colleagues at both institutions and several students for numerous suggestions and valuable criticism. Mrs. Joan Haugen painstakingly typed the manuscript, and Mrs. Christel Siahatgar prepared most of the extensive tables. Very special thanks go to these two ladies and to the author's wife for their help, patience, and encouragement.

SELECTED BIBLIOGRAPHY ON ATOMIC SPECTRA

- Condon, E. U., and G. H. Shortley, 1951. "The Theory of Atomic Spectra," Cambridge University Press, New York.
- Herzberg, G., 1937. "Atomic Spectra and Atomic Structure," Prentice-Hall, Englewood Cliffs, N.J.
- Judd, B. R., 1963. "Operator Techniques in Atomic Spectroscopy," McGraw-Hill Book Company, New York.
- Kuhn, H. G., 1962. "Atomic Spectra," Academic Press, New York.
- Slater, J. C., 1960. "Quantum Theory of Atomic Structure," McGraw-Hill Book Company, New York.
- White, H. E., 1934. "Introduction to Atomic Spectra," McGraw-Hill Book Company, New York.

Contents

Preface v

1 Classical radiation theory 1

- 1-1 Electromagnetic Equations 2
- 1-2 Fields from Moving Point Charges 4
- 1-3 Emission of Dipole Radiation 6
- 1-4 Absorption by Harmonic Oscillators 7
- 1-5 Radiation Damping 8
- 1-6 Scattering of Radiation 10
- 1-7 Optical Refractivity 11
- 1-8 Magnetic Radiation 12
- Problems 14
- Bibliography 15

2 Quantum theory of radiation 17

- 2-1 Quantum Theory of Particles 18
- 2-2 Quantum Theory of Fields 20
- 2-3 Quantum Theory of Particles and Fields 22
- 2-4 Density of Final States and Normalization of the Vector Potentials 25
- 2-5 Spontaneous Emission 26

2-6	Absorption	28
2-7	Induced Emission	29
2-8	Natural Line Broadening	37
2-9	Scattering of Radiation	35
2-10	Resonance Fluorescence	38
2-11	Optical Refractivity	40
	Problems	43
	Bibliography	43

3 Calculation of line oscillator strengths 45

3-1	Relative Line Strengths	47
3-2	Absolute Line Strengths	51
3-3	Coulomb Approximation	52
3-4	Hartree-Fock Calculations	53
3-5	Variational Calculations	54
3-6	Semiempirical Methods	55
3-7	Numerical Results	56
3-8	Sum Rules	58
	Problems	60
	Bibliography	60
	References	60

4 Line-broadening calculations 63

4-1	General Theory of Pressure Broadening	64
4-2	Classical Path Approximation	66
4-3	Impact Approximation	69
4-4	Quasi-static Approximation	72
4-5	Stark Broadening of Hydrogen Lines	74
4-6	Stark Broadening of Ionized-helium Lines	78
4-7	Stark Broadening of Neutral-helium Lines	81
4-8	Stark Broadening of Lines from Light and Medium Elements	86
4-9	Numerical Results for Stark Profiles and Stark-broadening Parameters	88
4-10	Asymptotic Wing Formulas for Stark-broadened Lines	92
4-11	Asymmetries and Shifts of Stark-broadened Hydrogen and Hydrogenic Lines	93
4-12	Resonance Broadening	95
4-13	Van der Waals Broadening	98
4-14	Doppler Broadening and Superposition of Broadening Mechanisms	101
	Problems	102
	Bibliography	102
	References	103

5 Calculation of continua 105

5-1	General Formulas for Photoelectric Cross Sections	107
5-2	Quantum-defect Method	109

5-3	Numerical Results for Continuous-absorption Cross Sections	112
5-4	Recombination Radiation and Bremsstrahlung	113
5-5	Negative-ion and Molecular Continua	119
5-6	Plasma Corrections	121
5-7	Advance of the Series Limits	124
	Problems	126
	References	127

6 Equilibrium relations 129

6-1	Thermodynamic-equilibrium Considerations	131
6-2	Boltzmann Factors and Saha Equations	134
6-3	Debye Theory of Coulomb Interactions in Plasmas	137
6-4	Reduction of Ionization Energies	139
6-5	Calculation of Partition Functions	140
6-6	LTE-Plasma Composition and Pressure	142
6-7	Validity of LTE for Highly Excited Levels in Time-independent and Homogeneous Plasmas	145
6-8	Validity of Complete LTE in Time-independent and Homogeneous Plasmas	150
6-9	Validity Criteria for LTE in Homogeneous Transient Plasmas	152
6-10	Validity Criteria for LTE in Inhomogeneous Stationary Plasmas	156
6-11	Corona-equilibrium Relations	159
	Problems	167
	References	168

7 Radiation from extended sources 169

7-1	Emission Coefficients of Hot Plasmas	170
7-2	Radiative Transfer in Collision-dominated LTE Plasmas	172
7-3	Emission from Optically Thin Cylindrical Sources	176
7-4	Emission from Optically Thick Cylindrical Sources in LTE	178
7-5	Source Functions and Scattering	180
7-6	Transient Problems	184
	Problems	188
	References	188

8 Radiative energy losses 191

8-1	Bremsstrahlung	193
8-2	Recombination Radiation	194
8-3	Line Radiation	196
8-4	Magnetic Radiation	199
	Problems	202
	References	202

9 Plasma light sources 203

- 9-1 Stabilized Arcs 205
- 9-2 Demixing in Arc Plasmas 206
- 9-3 Pressure Corrections in Arc Plasmas 207
- 9-4 Luminous Shock Tubes 210
- 9-5 Relaxation Phenomena in Shock Tubes 212
- 9-6 Deviations from Ideal Hydrodynamic Behavior in Luminous Shock Tubes 214
- 9-7 Electromagnetic Shock Tubes 215
- 9-8 Ovens and Hot-cathode Diodes 218
- 9-9 Gas Compressors 220
- 9-10 Non-LTE Plasma Light Sources 222
 - Problems 223
 - References 223

10 Optical instruments 227

- 10-1 Survey Spectrographs 229
- 10-2 High-speed-drum Spectrographs 230
- 10-3 Photoelectric Spectrographs 232
- 10-4 Monochromators 233
- 10-5 Vacuum-ultraviolet Spectrographs and Monochromators 235
- 10-6 Soft-x-ray Monochromators and Spectrographs 236
- 10-7 Interferometers 238
 - Problems 240
 - References 240

11 Radiation detectors 243

- 11-1 Photomultiplier Tubes for Visible and Near-uv Radiation 245
- 11-2 Photomultipliers for Vacuum-uv Radiation 246
- 11-3 Special High-gain Vacuum-uv Detectors 247
- 11-4 Ionization Chambers and Photon Counters for Vacuum-uv Radiation 249
- 11-5 Soft-x-ray Detectors 251
 - Problems 252
 - References 252

12 Radiation standards 255

- 12-1 Blackbody Radiators 257
- 12-2 Tungsten Ribbon Lamps 259
- 12-3 Carbon Arcs 261
- 12-4 Near-uv and Vacuum-uv Intensity Calibration 262

Problems	265
References	265

13 Temperature measurements 267

13-1	Relative Intensities of Lines from the Same Element and Ionization Stage	269
13-2	Relative Line Intensities of Subsequent Ionization Stages of the Same Element	272
13-3	Relative Line-to-Continuum Intensities	279
13-4	Relative Continuum Intensities	283
13-5	Intensities from Optically Thick Layers	288
13-6	Excitation and Ionization Rates	291
13-7	Doppler Profiles	293
	Problems	295
	References	295

14 Density measurements 297

14-1	Optical Refractivities	299
14-2	Stark Profiles	303
14-3	Absolute Line Intensities	307
14-4	Absolute Continuum Intensities	309
	Problems	312
	References	312

15 Measurements of atomic parameters 315

15-1	Oscillator Strengths	318
15-2	Continuum-absorption Coefficients	323
15-3	Stark Profiles	325
15-4	Collisional-rate Coefficients	331
	Problems	332
	References	333

Tables	336
--------	-----

List of Symbols	543
-----------------	-----

Solutions	549
-----------	-----

Index	575
-------	-----

Classical Radiation Theory

1

Most of the electromagnetic radiation emitted by gaseous plasmas stems from atomic processes. Plasma spectroscopy is therefore based on atomic and plasma physics and requires mainly an understanding of quantum theory and the statistical mechanics of ionized gases. In both fields the basic theory is well known. However, much detailed work remains to be done to elucidate the nature of numerous approximate treatments and to assess the ranges of applicability and the degree of approximation that has been achieved.

Because of the merging of different branches of physics, an unusually close interplay between theory and experiment is mandatory. In this respect, plasma spectroscopy tends to follow the example of astrophysics. Since the development of stars is decisively influenced by radiative energy transfer and since electromagnetic radiation is by far the most important carrier of information available to them, astronomers have developed

applications of the theory of radiation to these two problems, which predominate in plasma spectroscopy as well.

Astronomers have felt compelled to resume work on applications of quantum theory to atomic physics; plasma spectroscopists also must contribute to the improvement of our knowledge and command of atomic radiation theory. Besides methods for the determination of plasma parameters and the investigation of radiative transfer and energy loss problems, a third main objective of plasma spectroscopy is therefore to help in the establishment of a reliable system of atomic parameters.

The difficulties in applications of quantum mechanics are hardly ever of a basic nature. However, experiments are indispensable because precise calculations are often practically impossible, even with modern electronic computers, or because no reliable error estimate can be made for a completed calculation. Both to see more clearly the remaining problems and also to appreciate the power and reliability of plasma spectroscopy as a tool, it is best to follow the development of radiation theory, before the combination with statistical mechanics and more specific applications are taken up.

1-1 Electromagnetic equations

The rationalized mks system of units is used unless specifically stated otherwise. (However, working equations will usually be written in such a way that cgs units can be used as well.) The vacuum values of dielectric constant and magnetic permeability in the mks system are $\epsilon_0 = (4\pi \times 9 \times 10^9)^{-1} = 8.85 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2}$ and $\mu_0 = 4\pi \times 10^{-7} = 12.57 \times 10^{-7} \text{ W A}^{-1} \text{ m}^{-1}$. The newton is the unit of force ($1 \text{ N} = 10^5 \text{ dyn}$); the weber is the unit of magnetic flux ($1 \text{ W m}^{-2} = 10^4 \text{ G}$). The units of charge and current are coulomb and ampere, respectively, and one also has $(\epsilon_0 \mu_0)^{-1/2} = c \approx 3 \times 10^8 \text{ m/sec}$.

Maxwell's equations

$$\text{curl } \mathbf{E} + \dot{\mathbf{B}} = 0 \quad (1-1a)$$

$$\text{div } \mathbf{B} = 0 \quad (1-1b)$$

$$\text{curl } \mathbf{H} - \dot{\mathbf{D}} = \mathbf{j} \quad (1-1c)$$

$$\text{div } \mathbf{D} = \rho \quad (1-1d)$$

and Lorentz's force law

$$\mathbf{F} = \rho \mathbf{E} + \mathbf{j} \times \mathbf{B} \quad (1-2)$$

form the basis of classical radiation theory. If ρ is the density of all

charges, one has simply (in vacuum)

$$\mathbf{B} = \mu_0 \mathbf{H} \quad (1-3a)$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} \quad (1-3b)$$

Furthermore, the current density is for charges of one sign

$$\mathbf{j} = \rho \mathbf{v} \quad (1-4)$$

where \mathbf{v} is the mean velocity.

It is convenient to introduce vector and scalar potentials through

$$\mathbf{B} = \text{curl } \mathbf{A} \quad (1-5a)$$

$$\mathbf{E} = -\text{grad } \varphi - \dot{\mathbf{A}} \quad (1-5b)$$

This choice automatically fulfills the first two of Maxwell's equations. In addition, one postulates the following Lorentz condition

$$\text{div } \mathbf{A} = -\epsilon_0 \mu_0 \dot{\varphi} \quad (1-6)$$

Substitution into the last two of Maxwell's equations, with $\text{curl curl } \mathbf{A} = \text{grad div } \mathbf{A} - \nabla^2 \mathbf{A}$ (in cartesian coordinates), yields

$$\frac{1}{c^2} \ddot{\mathbf{A}} - \nabla^2 \mathbf{A} = \mu_0 \rho \mathbf{v} \quad (1-7a)$$

$$\frac{1}{c^2} \ddot{\varphi} - \nabla^2 \varphi = \frac{1}{\epsilon_0} \rho \quad (1-7b)$$

A special solution of these inhomogeneous wave equations is obtained from a generalization of the Coulomb potential as

$$\mathbf{A}(\mathbf{r}'', t) = \frac{\mu_0}{4\pi} \int \frac{\rho' \mathbf{v}'}{|\mathbf{r}'' - \mathbf{r}'|} d\tau' \quad (1-8a)$$

$$\varphi(\mathbf{r}'', t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho'}{|\mathbf{r}'' - \mathbf{r}'|} d\tau' \quad (1-8b)$$

where ρ' , \mathbf{v}' are to be taken at \mathbf{r}' and $t' = t - |\mathbf{r}'' - \mathbf{r}'|/c$, the integration being over all points characterized by \mathbf{r}' . That the "retarded" potentials in Eqs. (1-8a) and (1-8b) obey the wave equations (1-7a) and (1-7b) can be seen as follows: Any function of $t' = t - |\mathbf{r}'' - \mathbf{r}'|/c$ divided by $|\mathbf{r}'' - \mathbf{r}'|$ presents a spherical wave emerging from \mathbf{r}' , that is, is a solution of the homogeneous wave equation for $\mathbf{r}'' \neq \mathbf{r}'$. On substitution of the potentials into the wave equations, therefore, only points \mathbf{r}' near \mathbf{r}'' contribute to the left-hand side. The remaining integrals over a small sphere with radius r surrounding \mathbf{r}'' are of the order r^2 (times the local charge or current density) and therefore their second spatial derivatives are of the

4 Plasma spectroscopy

order 1. But the second time derivatives divided by r^2 vanish as

$$\frac{\omega^2 r^2}{c^2} = \left(\frac{2\pi r}{\lambda} \right)^2$$

where ω is a frequency characterizing the time variations of ρ or $\rho\mathbf{v}$, and λ is the wavelength of the corresponding radiation. Also, $\rho'(\mathbf{r}',t')$ and $\mathbf{v}'(\mathbf{r}',t')$ approach $\rho(\mathbf{r}'',t)$ and $\mathbf{v}(\mathbf{r}'',t)$, so that the remaining integrals become just the well-known static solutions which indeed satisfy Eqs. (1-7a) and (1-7b) without the terms involving the time derivative. That the potentials also obey Lorentz's condition [Eq. (1-6)] follows simply from the equation of continuity:

$$\text{div}(\rho\mathbf{v}) + \dot{\rho} = 0 \quad (1-9)$$

The retarded potentials are, of course, special solutions of the wave equations. To obtain the general solution, solutions of the homogeneous wave equations must be added. For the present purposes, one can assume that only the vector potential contains such solutions obeying the homogeneous wave equation and the simplified Lorentz condition, namely,

$$\frac{1}{c^2} \ddot{\mathbf{A}} - \nabla^2 \mathbf{A} = 0 \quad (1-10a)$$

$$\text{div} \mathbf{A} = 0 \quad (1-10b)$$

The corresponding fields $\mathbf{B} = \text{curl} \mathbf{A}$ and $\mathbf{E} = -\dot{\mathbf{A}}$ represent transverse waves, and the required solutions of the inhomogeneous wave equations are therefore superpositions of transverse waves to the fields derived from the retarded potentials in Eqs. (1-8a) and (1-8b).

1-2 Fields from moving point charges

In spectroscopy one is concerned with radiation whose wavelength is very much greater than the classical electron radius

$$r_0 = \frac{e^2}{4\pi\epsilon_0 mc^2} \approx 2.82 \times 10^{-5} \text{ \AA}$$

Therefore the structure of the electron is of no consequence; i.e., it can always be replaced by a point charge with total charge $e = \int \rho d\tau$ and mass m . Accordingly, the objective is to express fields and potentials in terms of total charge e and velocity \mathbf{v} of the electron and then to see under which conditions electromagnetic radiation is produced. For this purpose, one needs to study the fields produced by moving or, more precisely, by accelerated point charges.

Some care must be taken in the evaluation of the retarded potentials produced by such point charges. In general, the integral $\int \rho' d\tau'$ does not equal the total charge, because the retarded times are different for different volume elements. One may first consider the contribution to the total charge e for all points that are between radii r and $r + dr$, measured from the point at which one wants to know the potentials. To the charge $\rho'(\mathbf{r}', t') d\tau' = \rho' d\sigma dr$ actually present at time t' one must add a term that compensates for the difference of the charges streaming through the elements of surface $d\sigma$ at r and $r + dr$, respectively, in the times $t - t' = r/c$ and $t - t' - dt' = (r + dr)/c$. If \mathbf{r} is the vector from the field point to the source point, this difference is $-\rho' \mathbf{v}' \cdot \mathbf{r} d\sigma dr/cr$. That is,

$$de = \left(1 + \frac{\mathbf{v}' \cdot \mathbf{r}}{cr}\right) \rho' d\sigma dr$$

is the charge that must be assigned to $d\tau'$ in order to account for the total charge. Transformation from the volume element $d\tau' = d\sigma dr$ to the element of charge de in Eqs. (1-8a) and (1-8b) then yields Lienard's and Wiechert's retarded potentials for point charges,

$$\mathbf{A}(t) = \frac{\mu_0}{4\pi} \frac{e\mathbf{v}}{r + \mathbf{r} \cdot \mathbf{v}/c} \Big|_{t-r/c} \quad (1-11a)$$

$$\varphi(t) = \frac{1}{4\pi\epsilon_0} \frac{e}{r + \mathbf{r} \cdot \mathbf{v}/c} \Big|_{t-r/c} \quad (1-11b)$$

Here \mathbf{v} , \mathbf{r} , and $r = |\mathbf{r}|$ must all be taken at the retarded time $t - r/c$.

The fields produced by a moving point charge follow from Eqs. (1-5a) and (1-5b), finally, as

$$\mathbf{E} = -\frac{e}{4\pi\epsilon_0} \left\{ \left(1 - \frac{v^2}{c^2}\right) \left(\mathbf{r} + \frac{\mathbf{v}\mathbf{r}}{c}\right) - \frac{1}{c^2} \mathbf{r} \times \left[\left(\mathbf{r} + \frac{\mathbf{v}\mathbf{r}}{c}\right) \times \dot{\mathbf{v}} \right] \right\} \left(r + \frac{\mathbf{v} \cdot \mathbf{r}}{c}\right)^{-3} \quad (1-12a)$$

$$\mathbf{B} = \frac{\mathbf{E} \times \mathbf{r}}{rc} \quad (1-12b)$$

(For details of the derivation, see, for example, Heitler's book listed in the Bibliography at the end of the chapter.) At great distances only the term containing the acceleration $\dot{\mathbf{v}}$ is significant, because it decays as $1/r$, as compared with the $1/r^2$ dependence of the first term. The magnetic field is always transverse, but the electric field becomes purely transverse only at great distances, i.e., in the wave zone. There the Poynting vector is proportional to \mathbf{r}/r^3 , and the energy flux (radiation) is the same through any surface enclosing the charge.

1-3 Emission of dipole radiation

The fields from a system of point charges are simply obtained from the superposition of fields produced by single charges, because Maxwell's equations are linear. Usually one is interested in the radiation from a small number of closely spaced point charges e_k that are positioned at $\mathbf{r}_k = \bar{\mathbf{r}} + \mathbf{x}_k$, where $\bar{\mathbf{r}}$ is a vector denoting the position of the center of charge and \mathbf{x}_k the vector describing the displacement of charge k from this center. Assuming $|\bar{\mathbf{r}}| \gg |\mathbf{x}_k|$ and $v/c \ll 1$, one then derives from Eqs. (1=12a) and (1=12b) for the part of the field that vanishes only as $1/r$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \bar{\mathbf{r}} \times \frac{\bar{\mathbf{r}} \times \sum e_k \ddot{\mathbf{x}}_k}{c^2 |\bar{\mathbf{r}}|^3} \quad (1-13a)$$

$$\mathbf{B} = \frac{1}{4\pi\epsilon_0} \frac{\bar{\mathbf{r}} \times \sum e_k \ddot{\mathbf{x}}_k}{c^3 |\bar{\mathbf{r}}|^2} \quad (1-13b)$$

These fields are therefore proportional to the second time derivative of the electric dipole moment $\sum e_k \mathbf{x}_k$ associated with the system of charges. If the differences in the retarded times for the various charges are taken into account and all quantities are calculated to the first order in v/c , one obtains an additional term that is proportional to the quadrupole moment, etc. But in plasma spectroscopy one can practically always neglect these terms because emitting electrons only very rarely approach relativistic energies.

The Poynting vector corresponding to the dipole radiation field is

$$\begin{aligned} \mathbf{S} = \mathbf{E} \times \mathbf{H} &= \frac{-1}{(4\pi)^2 \epsilon_0} \frac{|\bar{\mathbf{r}} \times \sum e_k \ddot{\mathbf{x}}_k|^2 \bar{\mathbf{r}}}{c^3 |\bar{\mathbf{r}}|^5} \\ &= \frac{-1}{(4\pi)^2 \epsilon_0} \frac{|\sum e_k \ddot{\mathbf{x}}_k|^2 \bar{\mathbf{r}} \sin^2 \theta}{c^3 |\bar{\mathbf{r}}|^3} \end{aligned} \quad (1-14)$$

where θ is the angle between the direction of observation and the second derivative of the dipole moment. Since $\bar{\mathbf{r}}$ is a vector pointing toward the system of charges, the minus sign indicates that the Poynting vector is in a direction away from the system, which is as it should be. Integration of \mathbf{S} over a closed surface containing all charges k finally yields the radiated power

$$P_e = \frac{1}{6\pi\epsilon_0 c^3} |\sum e_k \ddot{\mathbf{x}}_k|^2 \quad (1-15)$$

which is the quantity of primary interest in quantitative spectroscopy.

The harmonic oscillator with $\sum e_k \mathbf{x}_k = e\mathbf{x}(t) = e\mathbf{x}_0 \cos \omega_0 t$ is the sim-

plest model of a radiating dipole. It emits an average power

$$P_e = \frac{e^2 \omega_0^4}{6\pi\epsilon_0 c^3} \overline{|x(t)|^2} = \frac{e^2 \omega_0^4}{12\pi\epsilon_0 c^3} |\mathbf{x}_0|^2 \quad (1-16)$$

According to Eq. (1-13a), the electric vector is at right angles to the direction of observation and in the plane containing this direction and that of \mathbf{x}_0 . The directional dependence of the emitted intensity is given by the factor $\sin^2 \theta$ in Eq. (1-14), with θ now being the angle between \mathbf{x}_0 and the radius vector. (Note also the very strong frequency dependence, which suggests that short-wavelength line radiation will be rather strong.)

1-4 Absorption by harmonic oscillators

Harmonic oscillators not only emit electromagnetic radiation but also may extract energy from incident waves. If the wave field is decomposed into Fourier components, the equation of motion for the oscillator becomes

$$\ddot{\mathbf{x}} + \omega_0^2 \mathbf{x} = \frac{e}{m} \mathbf{E}_\omega \cos(\omega t + \delta_\omega) \quad (1-17)$$

where δ_ω is the phase of the wave. Assuming that at $t = 0$ only the free oscillation is excited, one obtains the solution

$$\mathbf{x}(t) = \frac{e}{m} \mathbf{E}_\omega \frac{\cos(\omega t + \delta_\omega) - \cos(\omega_0 t + \delta_\omega)}{\omega_0^2 - \omega^2} + \mathbf{x}_0 \sin(\omega_0 t + \varphi) \quad (1-18)$$

The absorbed power follows from the rate of work done on the harmonic oscillator as

$$\begin{aligned} dP_a &= e\dot{\mathbf{x}}(t) \cdot \mathbf{E}_\omega \cos(\omega t + \delta_\omega) \\ &= \frac{e^2}{m} E_\omega^2 \left[-\frac{\omega}{\omega_0^2 - \omega^2} \sin(\omega t + \delta_\omega) + \frac{\omega_0}{\omega_0^2 - \omega^2} \sin(\omega_0 t + \delta_\omega) \right] \\ &\quad \times \cos(\omega t + \delta_\omega) + e\dot{\mathbf{x}}_0 \cdot \mathbf{E}_\omega \cos(\omega_0 t + \varphi) \cos(\omega t + \delta_\omega) \end{aligned} \quad (1-19)$$

In the time average, the term with $\omega/(\omega_0^2 - \omega^2)$ vanishes. If the phases δ_ω in the incoming light wave are random, the last term also disappears. Using well-known trigonometric formulas and again assuming the phases δ_ω to be random, the average absorbed power thus becomes

$$\begin{aligned} d\bar{P}_a &= \frac{e^2}{2m} E_\omega^2 \frac{\omega_0}{\omega_0^2 - \omega^2} \frac{1}{\tau} \int_0^\tau \sin[(\omega_0 - \omega)t] dt \\ &= \frac{e^2}{2m} E_\omega^2 \frac{\omega_0}{\omega_0 + \omega} \frac{1}{\tau} \frac{1 - \cos[(\omega_0 - \omega)\tau]}{(\omega_0 - \omega)^2} \end{aligned} \quad (1-20)$$