PARTIAL DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS

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Preface

The theory of partial differential equations has become one of the most important fields of study in mathematical analysis, mainly due to the frequent occurrence of partial differential equations in many branches of physics, engineering, and other sciences. The study of these equations has been intensive and extensive; as a result, several books on the subject have been published. Despite the number of excellent textbooks available, this book has been written to present an approach based mainly on the mathematical problems and their related solutions, and also to formulate a course appropriate for all students of the mathematical sciences. The primary concern, therefore, is not with the general theory, but to provide students with the fundamental concepts, the underlying principles, and the techniques and methods of solution of partial differential equations.

An attempt has been made to present a clear and concise exposition of the mathematics used in analyzing a variety of problems. With this in mind, the chapters are carefully arranged to enable students to view the material to be studied in an orderly perspective. Theorems, for example, those in the chapters on Fourier series and eigenvalue problems, are explicitly mentioned whenever possible to avoid confusion with their use in the development of principles of partial differential equations. A wide range of problems in mathematical physics, with various boundary conditions, has been included to improve student understanding.

This book is in part based on lectures given at Manhattan College. It is used by advanced undergraduate or beginning graduate students in applied mathematics, physics, engineering, and other sciences. The prerequisite for its study is a standard calculus sequence with elementary ordinary differential equations.

The first chapter is concerned mainly with an introduction to partial differential equations. The second chapter deals with mathematical models corresponding to physical events that yield the three basic types of partial differential equations. The third chapter constitutes a full account of the classification of second order equations with two independent variables, and in addition, illustrates the determination of the general solution for a class of relatively simple equations.

After attaining some knowledge of the characteristics of partial differential equations, the student may continue on to the Cauchy problem, the Hadamard example and the Riemann method for initial value problems, as presented in the fourth chapter. The fifth chapter contains a brief but thorough treatment of Fourier series, essential for the further study of partial differential equations.

Separation of variables is one of the simplest and most widely used method of solving partial differential equations. Its basic concept and the separability conditions necessary for its application are described in the sixth chapter, followed by some well-known problems of mathematical physics with a detailed analysis of each problem. In the seventh chapter, eigenvalue problems are treated in depth, building on their introduction in the preceding chapter. In addition, Green's function and its application to eigenvalue problems are developed briefly.

Boundary value problems and the maximum principle are presented in the eighth chapter whereas more involved higher dimensional problems and the eigenfunction method are treated in the ninth chapter. The tenth chapter deals with the basic concepts and the construction of the Green's function and its application to boundary value problems. In the final chapter the fundamental properties and the techniques of Fourier and Laplace transforms are introduced.

The chapters on mathematical models, Fourier series and eigenvalue problems are self-contained, hence these chapters can be omitted for those students who have prior knowledge of the subjects. The exercises are an integral part of the text and range from simple to more difficult problems. Answers to most exercises are given at the end of the book. For students wishing further insight into the subject matter, detailed references are listed in the Bibliography.

The author wishes to express sincere appreciation to his colleagues and the students who used the mimeographed edition of this book, and to Mr. John Adamczak for his kind assistance in the preparation of the answers. The author also wishes to thank Professor Arthur Schlissel for reading the first part of the original manuscript and for offering many helpful comments, and Professor Donald Gelman for reading the entire manuscript and for rendering most valuable comments and suggestions. The author also extends his profound gratitude to the reviewers for their constructive criticisms and suggestions and to the staff of American Elsevier for their kind help and cooperation. Finally, he wishes to express his heartfelt thanks to his wife Aye for her patience, understanding and encouragement necessary for completion of this book.

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CHAPTER 1

Introduction

1.1. Basic Concepts and Definitions

A differential equation that contains, in addition to the dependent variable and the independent variables, one or more partial derivatives of the dependent variable is called a *partial differential equation.* In general, it may be written in the form

$$f(x,y,\ldots,u,u_x,u_y,\ldots,u_{xx},u_{xy},\ldots) = 0 \tag{1.1.1}$$

involving several independent variables $x, y, \ldots$, an unknown function u of these variables and the partial derivatives[1] u_x, $u_y, \ldots$, u_{xx}, $u_{xy}, \ldots$ of the function. Here Eq. (1.1.1) is considered in a suitable domain D of the n-dimensional space R^n in the independent variables $x, y, \ldots$. We seek functions $u = u(x,y,\ldots)$ which satisfy Eq. (1.1.1) identically in D. Such functions, if they exist, are called *solutions* of Eq. (1.1.1). From these many possible solutions we attempt to select a particular one by introducing suitable additional conditions.

For instance,

$$\begin{aligned} uu_{xy} + u_x &= y \\ u_{xx} + 2yu_{xy} + 3xu_{yy} &= 4 \sin x \\ (u_x)^2 + (u_y)^2 &= 1 \\ u_{xx} - u_{yy} &= 0 \end{aligned} \tag{1.1.2}$$

are partial differential equations. The functions

$$u(x,y) = (x + y)^3$$

$$u(x,y) = \sin(x - y)$$

are solutions of the last equation of (1.1.2) as can easily be verified.

The *order* of a partial differential equation is the order of the highest-ordered partial derivative appearing in the equation. For example,

$$u_{xx} + 2xu_{xy} + u_{yy} = e^y$$

[1] Subscripts on dependent variables denote differentiations, e.g.

$$u_x = (\partial u/\partial x) \qquad u_{xy} = (\partial^2 u/\partial y \partial x)$$

is a second-order partial differential equation, and

$$u_{xxy} + xu_{yy} + 8u = 7y$$

is a third-order partial differential equation.

A partial differential equation is said to be *linear* if it is linear in the unknown function and all its derivatives with coefficients depending only on the independent variables; it is said to be *quasilinear* if it is linear in the highest-ordered derivative of the unknown function. For example, the equation

$$yu_{xx} + 2xyu_{yy} + u = 1$$

is a second-order linear partial differential equation whereas

$$u_x u_{xx} + xuu_y = \sin y$$

is a second order quasilinear partial differential equation. The equation which is not linear is called a *nonlinear* equation.

In this book we shall be primarily concerned with linear second-order partial differential equations frequently arising in problems of mathematical physics. The most general second-order linear partial differential equation in n independent variables has the form

$$\sum_{i,j=1}^{n} A_{ij} u_{x_i x_j} + \sum_{i=1}^{n} B_i u_{x_i} + Fu = G \tag{1.1.3}$$

where we assume without loss of generality that $A_{ij} = A_{ji}$. We also assume that B_i, F, and G are functions of the n independent variables x_i.

If G is identically zero, the equation is said to be *homogeneous*; otherwise it is *nonhomogeneous*.

The general solution of an ordinary differential equation of nth order is a family of functions depending on n independent arbitrary constants. In the case of partial differential equations the general solution depends on arbitrary functions rather than arbitrary constants. To illustrate this, we consider the second-order equation

$$u_{xy} = 0$$

If we integrate this equation with respect to y, holding x fixed, we obtain

$$u_x(x,y) = f(x)$$

A second integration, this time with respect to x while y is held fixed, yields

$$u(x,y) = g(x) + h(y)$$

where $g(x)$ and $h(y)$ are arbitrary functions.

Suppose u is a function of three variables, x, y, and z. Then for the equation

$$u_{yy} = 2$$

one finds the general solution

$$u(x,y,z) = y^2 + yf(x,z) + g(x,z)$$

where f and g are arbitrary functions of two variables x and z.

We recall that in the case of ordinary differential equations, the first task is to ascertain a general solution, and then the particular solution is determined by finding the values of arbitrary constants from the prescribed conditions. But, for partial differential equations, selecting a particular solution satisfying the supplementary conditions from the general solution of a partial differential equation may be as difficult as, or even more difficult than, the problem of finding the general solution itself. This is so because the general solution of a partial differential equation involves arbitrary functions; the specialization of such a solution to the particular form which satisfies supplementary conditions requires the determination of these arbitrary functions, rather than merely the determination of constants.

For linear homogenous ordinary differential equations of order n a linear combination of n linearly independent solutions is a solution. Unfortunately, this is not true, in general, in the case of partial differential equations. This is due to the fact that the solution space of every homogenous linear partial differential equation is infinite dimensional. For example, the partial differential equation

$$u_x - u_y = 0 \tag{1.1.4}$$

can be transformed into the equation

$$2u_\eta = 0$$

by the transformation of variables

$$\xi = x + y$$
$$\eta = x - y$$

The general solution is

$$u(x,y) = f(x + y)$$

where $f(x + y)$ is an arbitrary function and is everywhere differentiable. From this it follows that each of the functions

$$\left.\begin{array}{l}(x+y)^n\\ \sin n(x+y)\\ \cos n(x+y)\\ \exp n(x+y)\end{array}\right\} \quad \text{for } n = 1, 2, 3, \ldots$$

is a solution of Eq. (1.1.4), and it is evident that these functions are linearly independent. The fact, that a simple equation such as (1.1.4) yields infinitely many solutions, is an indication of an added difficulty which must be overcome in the study of partial differential equations. Thus, we generally prefer to determine directly the particular solution satisfying prescribed supplementary conditions.

1.2. Linear Operators

This section will be devoted to a brief discussion of linear operators which are often encountered in the theory of partial differential equations.

An operator is a mathematical rule which when applied to a function produces another function. For example, in the expressions

$$Lu = \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^3 u}{\partial y^3}$$

$$Mu = \frac{\partial^2 u}{\partial x^2} - x^2 \frac{\partial^2 u}{\partial y^2}$$

$$L = \frac{\partial}{\partial x} + \frac{\partial^2}{\partial x \partial y} + \frac{\partial^3}{\partial y^3}$$

$$M = \frac{\partial^2}{\partial x^2} - x^2 \frac{\partial^2}{\partial y^2}$$

are called differential operators. There are other types of operators, such as

$$P[u] = \int_a^b u(x,\tau)F(\tau,y)\,d\tau \qquad a, b \text{ are constants}$$

$$Q[u] = u(x,c) + u_x(x,c) \qquad c \text{ is a constant}$$

The operator P is an integral operator and the operator Q is an operator which transforms the function u of two variables x and y into the function $Q[u]$ of one variable x.

Two differential operators are said to be equal if, and only if, the same result is produced when each operates upon the function u, that is $A = B$ if, and only if,

$$A[u] = B[u] \tag{1.2.1}$$

for the function u. The function u must be sufficiently differentiable.

The sum of two differential operators A and B is defined as

$$(A + B)u = A[u] + B[u] \tag{1.2.2}$$

for the function u.

The product of two differential operators A and B is the operator which produces the same result as is obtained by the successive operations of the operators A and B on the function u, that is,

$$AB[u] = A(B[u]) \tag{1.2.3}$$

Differential operators satisfy the following:

(1) The commutative law of addition:

$$A + B = B + A \tag{1.2.4}$$

(2) The associative law of addition:

$$(A + B) + C = A + (B + C) \tag{1.2.5}$$

(3) The associative law of multiplication:

$$(AB)C = A(BC) \tag{1.2.6}$$

(4) The distributive law of multiplication with respect to addition:

$$A(B + C) = AB + AC \tag{1.2.7}$$

(5) The commutative law of multiplication:

$$AB = BA \tag{1.2.8}$$

holds only for differential operators with constant coefficients.

EXAMPLE 2.1. Let $A = \partial^2/\partial x^2 + x\partial/\partial y$ and $B = \partial^2/\partial y^2 - y\partial/\partial y$

$$B[u] = \partial^2 u/\partial y^2 - y\partial u/\partial y$$

$$\begin{aligned} AB[u] &= \left(\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial y}\right)\left(\frac{\partial^2 u}{\partial y^2} - y\frac{\partial u}{\partial y}\right) \\ &= \frac{\partial^4 u}{\partial x^2 \partial y^2} - y\frac{\partial^3 u}{\partial x^2 \partial y} + x\frac{\partial^3 u}{\partial y^3} - xy\frac{\partial^2 y}{\partial y^2} - x\frac{\partial u}{\partial y} \\ BA[u] &= \left(\frac{\partial^2}{\partial y^2} - y\frac{\partial}{\partial y}\right)\left(\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial y}\right) \\ &= \frac{\partial^4 u}{\partial y^2 \partial x^2} + x\frac{\partial^3 u}{\partial y^3} - y\frac{\partial^3 u}{\partial y \partial x^2} - xy\frac{\partial^2 u}{\partial y^2} \end{aligned}$$

thus $AB[u] \neq BA[u]$ for $x \neq 0$.

We define *linear operators* having the following properties:

(1) A constant c may be taken outside the operator:

$$L[cu] = cL[u]$$

(2) The operator operating on the sum of two functions gives the sum of the operator operating on the individual functions:

$$L[u + v] = L[u] + L[v]$$

Properties (1) and (2) may be combined to express

$$L[au + bv] = aL[u] + bL[v] \tag{1.2.9}$$

where a and b are constants.

Now let us consider a linear second-order partial differential equation. In the case of two independent variables, such an equation takes the form

$$\begin{aligned} A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} + D(x,y)u_x \\ + E(x,y)u_y + F(x,y)u = G(x,y) \end{aligned} \tag{1.2.10}$$

where the coefficients A, B, C, D, E, F are functions of variables x and y, and $G(x,y)$ is the nonhomogenous term.

If we take the linear differential operator L to be

$$L = A\frac{\partial^2}{\partial x^2} + B\frac{\partial^2}{\partial x \partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y} + F$$

then the differential equation (1.2.10) may be written in the form

$$L[u] = G \tag{1.2.11}$$

Very often the square bracket is omitted and one simply writes

$$Lu = G$$

1.3. Mathematical Problems

A problem consists of finding an unknown function of a partial differential equation satisfying appropriate supplementary conditions. These conditions may be initial and/or boundary conditions. For example

P.D.E.	$u_t - u_{xx} = 0$	$0 < x < l$	$t > 0$
I.C.	$u(x,0) = \sin x$	$0 \leqslant x \leqslant l$	
B.C.	$u(0,t) = 0$	$t \geqslant 0$	
B.C.	$u(l,t) = 0$	$t \geqslant 0$	

is a problem which consists of a partial differential equation and three supplementary conditions. The equation describes the heat conduction in a rod of length l. The last two conditions are called the *boundary conditions* which describe the function at two prescribed boundary points. The first condition is known as the *initial condition* which prescribes the unknown function $u(x,t)$ throughout the given region at some initial time t, in this case $t = 0$. This problem is known as the *initial-boundary value problem*. Mathematically speaking, the time and the space coordinates are regarded as some independent variables. In this respect, the initial condition is merely a point prescribed on the t-axis and the boundary conditions are prescribed, in this case, as two points on the x-axis. Initial conditions are usually prescribed at a certain time $t = t_0$ or $t = 0$, but it is not customary to consider the other end point of a given time interval.

In many cases, in addition to prescribing the unknown function, other conditions such as their derivatives are specified on the boundary.

In considering the problem of unbounded domain, the solution can be determined uniquely by prescribing initial conditions only. The problem is called the initial value problem.[2] The solution of such a problem may be interpreted physically as the solution unaffected by the boundary conditions at infinity. Later we shall discuss problems with boundedness conditions on the behavior of solutions at infinity.

A mathematical problem is said to be properly posed if it satisfies the following requirements:

(1) Existence: There is at least one solution.
(2) Uniqueness: There is at most one solution.
(3) Stability: The solution depends continuously on the data.

The first requirement is an obvious logical condition, but we must keep in mind that we cannot simply state that the mathematical problem has a solution just because the physical problem has a solution. The same can be said about the uniqueness requirement. The physical problem may have a unique solution but the mathematical problem may have more than one solution.

The last requirement is a necessary condition. In practice, small errors occur in the process of measurements. Thus for the mathematical problem to represent a physical phenomenon a small variation of the given data should lead to at most a small change in the solution.

1.4. Superposition

A linear partial differential equation has the form

$$L[u] = G$$

We may also express supplementary conditions using the operator notation. For instance, we may define

$$[u_t]_{x=0} = M_i[u]$$
$$[u]_{x=l} = M_j[u]$$

where the M operators are linear operators representing supplementary conditions. The initial-boundary value problem may thus be written as

[2] A mathematical definition will be given in Chapter 4.

$$\begin{aligned} L[u] &= G \\ M_1[u] &= g_1 \\ &\cdot \\ &\cdot \\ &\cdot \\ M_n[u] &= g_n \end{aligned} \tag{1.4.1}$$

where the first equation is a linear partial differential equation and the others are linear initial or boundary conditions. For example, the initial-boundary value problem

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= G(x,t) \qquad 0 < x < l \qquad t > 0 \\ u(x,0) &= g_1(x) \qquad 0 \leqslant x \leqslant l \\ u_t(x,0) &= g_2(x) \qquad 0 \leqslant x \leqslant l \\ u(0,t) &= g_3(t) \qquad t \geqslant 0 \\ u(l,t) &= g_4(t) \qquad t \geqslant 0 \end{aligned} \tag{1.4.2}$$

may be written in the form

$$\begin{aligned} L[u] &= G \\ M_1[u] &= g_1 \\ M_2[u] &= g_2 \\ M_3[u] &= g_3 \\ M_4[u] &= g_4 \end{aligned} \tag{1.4.3}$$

where g_i are the prescribed functions and the subscripts on operators are assigned arbitrarily.

We consider the problem (1.4.1). Let

$$u = v + w$$

where v is the particular integral of (1.4.1), that is

$$L[v] = G$$

Because of the linearity of the equation, we have

$$L[u] = L[v] + L[w] = G$$

so that

$$L[w] = 0$$

Thus we may state that the solution of a given partial differential equation can be presented as the sum of a particular solution and a solution of the "associated homogenous equation."