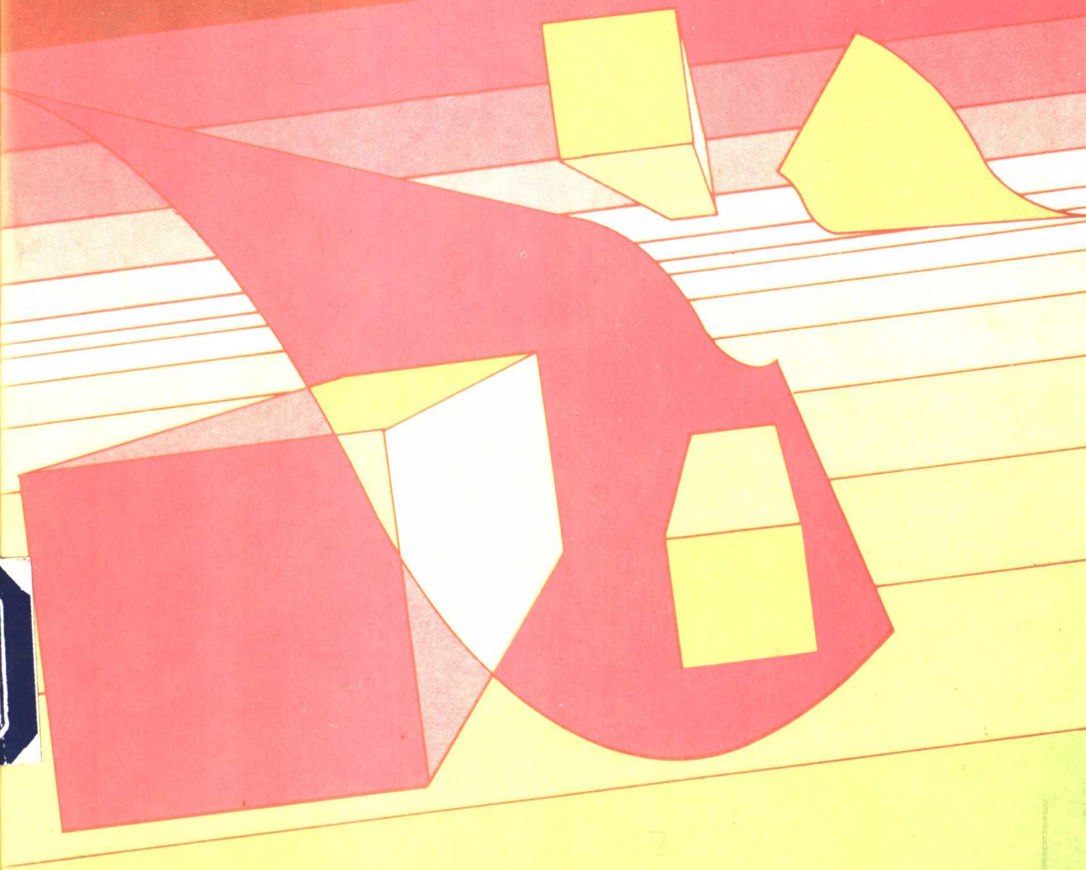
Ellis Horwood Series
MATHEMATICS AND ITS APPLICATIONS

THEORY AND APPLICATIONS OF LINEAR DIFFERENTIAL AND DIFFERENCE EQUATIONS

a systems approach in engineering

R.M. JOHNSON



THEORY AND APPLICATIONS OF LINEAR DIFFERENTIAL AND DIFFERENCE EQUATIONS:
A Systems Approach in Engineering

R. M. JOHNSON, B.Sc., F.I.M.A.
Senior Lecturer, Department of Mathematics and Computing
Paisley College of Technology

ELLIS HORWOOD LIMITED
Publishers · Chichester
Halsted Press: a division of
JOHN WILEY & SONS
New York · Chichester · Brisbane · Toronto

First published in 1984 by
ELLIS HORWOOD LIMITED
Market Cross House, Cooper Street, Chichester, West Sussex, PO19 1EB, England

The publisher's colophon is reproduced from James Gillison's drawing of the ancient Market Cross, Chichester.

Distributors:
Australia, New Zealand, South-east Asia:
Jacaranda-Wiley Ltd., Jacaranda Press,
JOHN WILEY & SONS INC.,
G.P.O. Box 859, Brisbane, Queensland 40001, Australia

Canada:
JOHN WILEY & SONS CANADA LIMITED
22 Worcester Road, Rexdale, Ontario, Canada.

Europe, Africa:
JOHN WILEY & SONS LIMITED
Baffins Lane, Chichester, West Sussex, England.

North and South America and the rest of the world:
Halsted Press: a division of
JOHN WILEY & SONS
605 Third Avenue, New York, N.Y. 10016, U.S.A.

British Library Cataloguing in Publication Data
Johnson, R. M. (Roy Michael)
Theory and applications of linear differential and difference equations. –
(Ellis Horwood series in mathematics and its applications)
1. Engineering mathematics 2. Differential equations 3. Difference equations
I. Title
620'.001'515354 TA347.D45

Library of Congress Card No. 84-12990

ISBN 0-85312-670-4 (Ellis Horwood Limited – Library Edn.)
ISBN 0-85312-802-2 (Ellis Horwood Limited – Student Edn.)
ISBN 0-470-20106-1 (Halsted Press)

Printed in Great Britain by The Camelot Press, Southampton.

Author's preface

This textbook provides a compact treatment of linear differential equations and linear difference equations using transform techniques. It is written as a textbook for engineering, science, computer science, and applied mathematics students in universities and polytechnics. Additionally it will provide a useful starting point for those who require to study more advanced texts, particularly in the field of signal analysis. Students on mathematics courses will find in the book an appreciation of how transform theory is applied in engineering situations.

The level of presentation is appropriate to second year degree mathematics students or to readers who are not specialists in mathematics, such as engineers and scientists, who study continuous and discrete linear systems. Such readers will require a mathematical background approximately equivalent to first year university mathematics.

The transform techniques used in the book are developed to encourage readers to think in terms of transfer functions and block diagrams rather than equations. An important relationship between the transform variables and frequency is established, and system stability is considered. Examples are chosen from the fields of Electrical Engineering, Mechanical Engineering, Civil Engineering, and Control Engineering. The book is probably unique in the way it uses frequency domain analysis to underline the similarities between differential equations and difference equations.

The development of microelectronic technology has given new emphasis to the analysis and filtering of discrete signals. The chapter on digital filters will be particularly useful to those whose background has been mainly in continuous systems.

PART I (Chapters 1–4). After a brief introduction to Fourier series, the Laplace transform is defined and applied to the solution of linear differential equations. Block diagram notation and transfer functions are introduced, and the language of control system analysis is used. Special emphasis is placed on the frequency response function and its application in the study of steady-state oscillations. An introduction is given to the concept of analog filtering and to the use of Bode diagrams. Chapter 4 considers differential equations with piecewise continuous forcing functions and serves as an introduction to sampling devices.

PART II (Chapters 5–8). Consideration of sampling devices leads to the definition of the z-transform which is then applied to the solution of linear difference equations. Again transfer functions are defined and block diagram notation is used. The use of transfer functions allows the similarities between differential equations and difference equations to be highlighted. In particular the methods for obtaining a system's steady-state output from its transfer function are compared. z-transform techniques are applied to simple sampled-data systems and to digital systems with reconstructed outputs. In the final chapter digital filters are introduced and simple design algorithms are established so that the performance of a given analog filter may be copied.

The "dot" notation, $\dot{x} \equiv dx/dt$, is used throughout the book; other notations are defined as they occur in the text. All system inputs are taken to be zero for $t < 0$.

The author is indebted to the many students at Paisley College who, in recent years, have been on the receiving end of much of the material contained in this textbook.

Particular thanks are expressed to Madeleine Stafford for the considerable task of typing and correcting the manuscript.

Finally I am grateful to the Series Editor, Professor G. Bell, to the publishers' referee, and to the staff of Ellis Horwood Limited for their valuable assistance and encouragement.

R. M. Johnson

Paisley 1984

Contents

PART 2: DISCRETE SYSTEMS

Part 1
Continuous systems

CHAPTER 1

An approach to the Laplace transform

1.1 INTRODUCTION

The Laplace transform is a powerful mathematical tool for problems arising from the study of continuous systems. The term 'continuous systems' is taken to imply systems which can be modelled by ordinary differential equations, for example

(i) a control system which positions a missile fin to achieve a certain lateral acceleration,
(ii) a crane where the position of the load is controlled by the application of hydraulic motors,
(iii) a structure subject to vibration.

The variables in these examples, position, acceleration, pressure, force, displacement, are *continuous* variables which can take any value within some specified range.

In Part 1 of this book we will apply Laplace transforms to *linear* continuous systems, that is systems described by linear differential equations. This can be done by accepting the mathematical definition of a Laplace transform as a starting point and turning directly to Chapter 2. This first chapter approaches the idea of a Laplace transform by considering the frequency characteristics of a function of time, and attempts to show the important relationship between the Laplace variable and frequency.

1.2 THE FOURIER SERIES OF A PERIODIC FUNCTION

A periodic function $f(t)$ satisfying certain conditions may be expressed as an infinite series which is a linear combination of sine and cosine functions whose frequencies are multiples of the fundamental frequency $\omega_0 = \frac{2\pi}{L}$, where L is the period of $f(t)$. The infinite series is known as the **Fourier series expansion** of $f(t)$ and takes the form

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t). \tag{1.1}$$

where the constants a_n and b_n are given by

$$a_n = \frac{\omega_0}{\pi}\int_{-\pi/\omega_0}^{\pi/\omega_0} f(t)\cos n\omega_0 t\,\mathrm{d}t. \tag{1.2}$$

$$b_n = \frac{\omega_0}{\pi}\int_{-\pi/\omega_0}^{\pi/\omega_0} f(t)\sin n\omega_0 t\,\mathrm{d}t. \tag{1.3}$$

$$\left(\text{note that } \frac{\pi}{\omega_0} = \frac{L}{2}\right)$$

Assuming that the Fourier series expansion exists for a given function $f(t)$, then equations (1.2) and (1.3) follow immediately from the orthogonality of the functions $\{\sin n\omega_0 t, \cos n\omega_0 t\}$ over the interval $\left[-\frac{\pi}{\omega_0}, \frac{\pi}{\omega_0}\right]$. That is, when $n \neq m$, m and n integers

$$\int_{-\pi/\omega_0}^{\pi/\omega_0} \cos n\omega_0 t \cos m\omega_0 t\,\mathrm{d}t = 0$$

$$\int_{-\pi/\omega_0}^{\pi/\omega_0} \sin n\omega_0 t \sin m\omega_0 t\,\mathrm{d}t = 0,$$

and for all integers m, n

$$\int_{-\pi/\omega_0}^{\pi/\omega_0} \sin n\omega_0 t \cos n\omega_0 t\,\mathrm{d}t = 0.$$

Sufficient conditions for the existence of the series (1.1) are that the function $f(t)$ is bounded and has a finite number of discontinuities and a finite number of maxima and minima in the interval $\left[-\frac{\pi}{\omega_0}, \frac{\pi}{\omega_0}\right]$. If $t = t_1$ is a point where $f(t)$ is continuous then the series converges to $f(t_1)$; if $t = t_2$ is a point where $f(t)$ is discontinuous then the series converges to $\frac{1}{2}\{f(t_2^+) + f(t_2^-)\}$, where $t_2 - \varepsilon < t_2^- < t_2 < t_2^+ < t_2 + \varepsilon$ for arbitrarily small ε. For a more detailed treatment of Fourier series see Kreyszig (1979).

Example 1.1
Obtain the Fourier series expansion of the function

$$f(t) = \begin{cases} 0, & -\pi \leqslant t < 0 \\ 1, & 0 \leqslant t < \pi \end{cases}, \quad f(t+2\pi) = f(t).$$

The function is shown in Fig. 1.1.

The period of the function is $L = 2\pi$, therefore $\omega_0 = 1$.

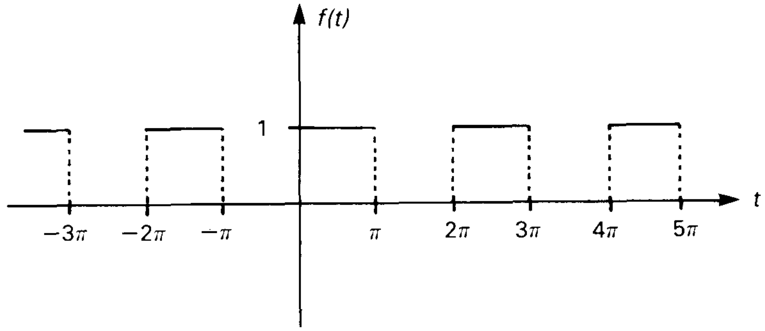


Fig. 1.1

Equations (1.2) and (1.3) give

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t \\ &= \frac{1}{\pi}\int_{0}^{\pi} \mathrm{d}t \\ &= 1 \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,\mathrm{d}t \quad \text{for} \quad n = 1, 2, 3, \ldots, \\ &= \frac{1}{\pi}\int_{0}^{\pi} \cos nt\,\mathrm{d}t \\ &= 0 \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,\mathrm{d}t \quad \text{for} \quad n = 1, 2, 3, \ldots, \\ &= \frac{1}{\pi}\int_{0}^{\pi} \sin nt\,\mathrm{d}t \\ &= \begin{cases} 2/\pi n, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even.} \end{cases} \end{aligned}$$

Substituting these results into equation (1.1) gives the required series,

$$f(t) = 0.5 + \frac{2}{\pi}\left\{\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \ldots\right\}.$$

Note that the above series converges to 1 for $0 < t < \pi$, converges to 0 for

$-\pi < t < 0$, and converges (clearly) to 0.5 when $t = 0$. For example, putting $t = \frac{\pi}{2}$,

$$1 = 0.5 + \frac{2}{\pi}\left\{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right\},$$

that is, $\sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)} = \frac{\pi}{4}$.

Example 1.2

The function $f(t) = t$ is defined in the interval $0 \leqslant t \leqslant l$. Obtain an infinite series expansion of $f(t)$ of the form

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{l}t, \quad 0 \leqslant t \leqslant l.$$

Hence show that

$$\sum_{r=0}^{\infty} \frac{1}{(2r+1)^2} = \frac{\pi^2}{8}.$$

We consider the *even* function $f_E(t)$ defined as follows

$$f_E(t) = \begin{cases} t, & 0 \leqslant t < l \\ -t, & -l \leqslant t < 0 \end{cases}, \quad f_E(t+2l) = f_E(t).$$

The function $f_E(t)$, shown in Fig. 1.2(a), is a periodic function of period $2l$ and coincides with the given function $f(t)$ in the interval $0 \leqslant t \leqslant l$. Since it is an even function, $f_E(t)$ will have a Fourier series expansion which contains only cosine terms. That is,

$$f_E(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t, \quad \omega_0 = \frac{\pi}{l}.$$

Therefore the function $f(t)$ will be represented by this series in the interval $0 \leqslant t \leqslant l$. ($f_E(t)$ is continuous for all t.)

Equation (1.2) gives

$$a_0 = \frac{1}{l}\int_{-l}^{l} f_E(t)\,dt$$

$$= \frac{2}{l}\int_{0}^{l} t\,dt, \text{ since } f_E(t) \text{ is even,}$$

$$= l$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(t)\cos\frac{n\pi}{l}t\,dt, \quad n = 1, 2, 3, \ldots$$

$$= \frac{2}{l}\int_0^l t\cos\frac{n\pi}{l}t\,dt$$

$$= \frac{2}{l}\left\{\frac{l^2}{n\pi}\sin n\pi + \frac{l^2}{n^2\pi^2}(\cos n\pi - 1)\right\}.$$

Therefore $a_n = \begin{cases} 0, \; n = 2, 4, 6, \ldots \\ \dfrac{-4l}{n^2\pi^2}, \; n = 1, 3, 5, \ldots \end{cases}$

and $f(t) = \dfrac{l}{2} - \dfrac{4l}{\pi^2}\left\{\cos\dfrac{\pi}{l}t + \dfrac{1}{3^2}\cos\dfrac{3\pi}{l}t + \dfrac{1}{5^2}\cos\dfrac{5\pi}{l}t + \ldots\right\}.$

Setting $t = 0$ gives

$$0 = \frac{l}{2} - \frac{4l}{\pi^2}\left\{1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right\},$$

that is, $\displaystyle\sum_{r=0}^{\infty}\frac{1}{(2r+1)^2} = \frac{\pi^2}{8}.$

Note that if we require a *sine* series for $f(t) = t, 0 \leqslant t \leqslant l$, then it is necessary to consider the periodic *odd* function,

$$f_o(t) = t, \; -l \leqslant t < l, \; f_o(t+2l) = f_o(t).$$

The Fourier series expansion of $f_o(t)$ contains only sine terms, that is, $a_n = 0$ for $n = 0, 1, 2, 3, \ldots$. (See Fig. 1.2(b) and Problem 3.)

Problems

(1) Obtain the Fourier series expansion of the function

$$f(t) = \begin{cases} \pi + t, & -\pi \leqslant t < 0 \\ \pi - t, & 0 \leqslant t < \pi \end{cases}, \; f(t+2\pi) = f(t).$$

(2) The output of a half-wave rectifier is given by

$$f(t) = \begin{cases} E\sin\dfrac{2\pi}{T}t, \; 0 \leqslant t < \dfrac{T}{2} \\ 0, \; \dfrac{T}{2} < t < T \end{cases}, f(t+T) = f(t).$$

Show that $f(t) = \dfrac{E}{\pi} + \dfrac{E}{2}\sin\dfrac{2\pi}{T}t - \dfrac{2E}{\pi}\left\{\dfrac{\cos\frac{4\pi}{T}t}{1.3} + \dfrac{\cos\frac{8\pi}{T}t}{3.5} + \ldots\right\}.$

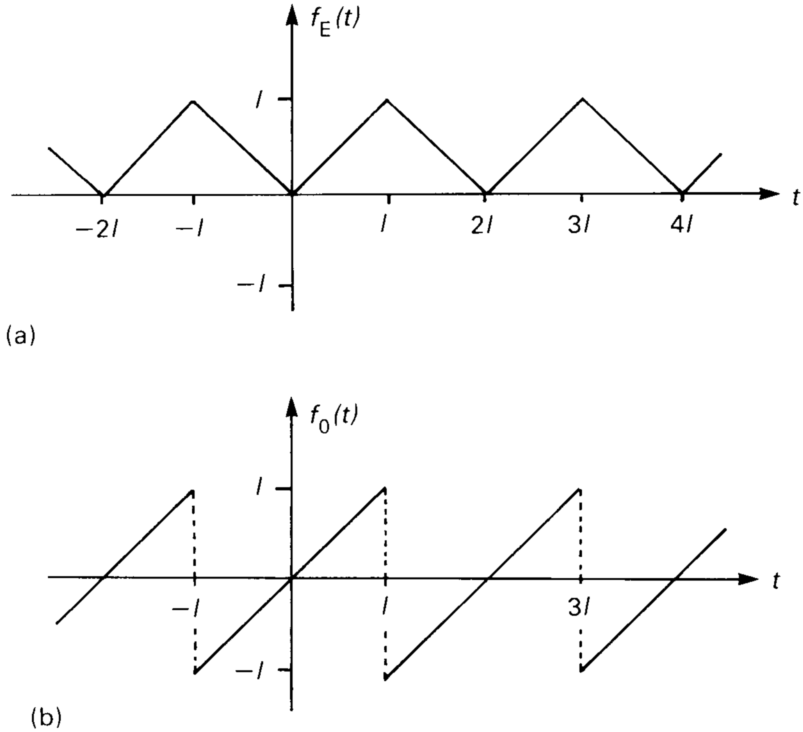


Fig. 1.2 (a) Even extension of $f(t)$, (b) Odd extension of $f(t)$.

(3) The function $f(t) = t$ is defined on the interval $0 \leqslant t \leqslant l$. Obtain an infinite series expansion of $f(t)$ in the form

$$f(t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} t.$$

(4) By consideration of the Fourier series for $f(t) = t^2, -\pi \leqslant t \leqslant \pi$, show that

(i) $\dfrac{\pi^2}{6} = 1 + \dfrac{1}{4} + \dfrac{1}{9} + \dfrac{1}{16} + \ldots$

(ii) $\dfrac{\pi^2}{12} = 1 - \dfrac{1}{4} + \dfrac{1}{9} - \dfrac{1}{16} + \ldots.$

(5) Obtain the Fourier series for $f(t) = \mathrm{e}^{-a|t|}, a > 0, -L \leqslant t \leqslant L$ and deduce that

$$\sum_{r=1}^{\infty} \left\{a^2 + (2r-1)^2 \frac{\pi^2}{L^2}\right\}^{-1} = \frac{L}{4a}\left\{\frac{1-\mathrm{e}^{-aL}}{1+\mathrm{e}^{-aL}}\right\}.$$

Answers to problems

(1) $\frac{\pi}{2}+\frac{4}{\pi}\left\{\cos t+\frac{1}{9}\cos 3t+\frac{1}{25}\cos 5t+\ldots\right\}$

(3) $\frac{2l}{\pi}\left\{\sin\frac{\pi}{l}t-\frac{1}{2}\sin\frac{2\pi}{l}t+\frac{1}{3}\sin\frac{3\pi}{l}t-\ldots\right\},\quad 0\leqslant t<l.$

(5) $\frac{1}{aL}(1-e^{-aL})+\frac{2a}{L}\sum_{n=1}^{\infty}(1-(-1)^n e^{-aL})\left(a^2+\frac{n^2\pi^2}{L^2}\right)^{-1}\cos\frac{n\pi}{L}t.$

1.3 COMPLEX FORM OF FOURIER SERIES

The sequence of numbers $\{a_0, a_1, a_2, \ldots, b_1, b_2, \ldots\}$ is an alternative way of defining the function $f(t)$ since given these numbers we can reconstruct the function using the Fourier series, equation (1.1). Writing the series as

$$f(t)=\frac{1}{2}a_0+\sum_{n=1}^{\infty} r_n\cos(n\omega_0 t+\phi_n)$$

where $r_n=\sqrt{(a_n{}^2+b_n{}^2)}$, $n=1, 2, 3, \ldots$, each number r_n provides a measure of the contribution to the periodic function $f(t)$ from the frequency $n\omega_0$. This information can be presented graphically in the form of a **Discrete amplitude spectrum**, see Fig. 1.3. In particular, the discrete amplitude spectrum for Example 1.1 is shown in Fig. 1.4.

An alternative, and more usual, representation of the discrete amplitude spectrum is related to the *complex form* of the Fourier series expansion.

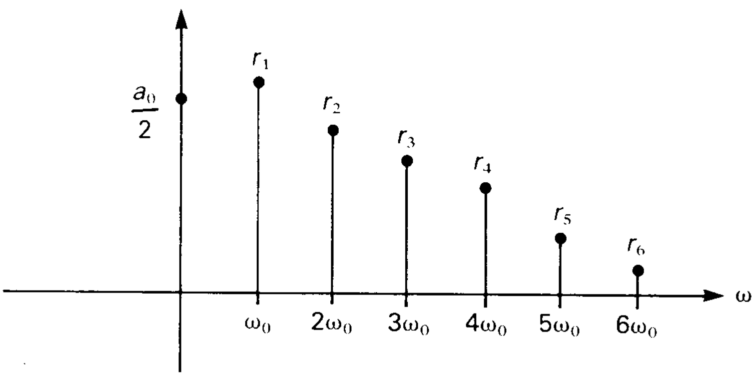


Fig. 1.3

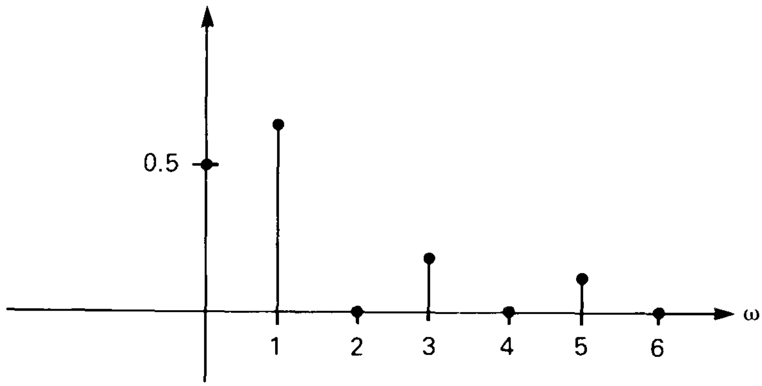


Fig. 1.4

Making the substitutions

$$\cos n\omega_0 t = \frac{1}{2}\Big(\exp(jn\omega_0 t) + \exp(-jn\omega_0 t)\Big)\dagger$$

$$\sin n\omega_0 t = \frac{1}{2j}\Big(\exp(jn\omega_0 t) - \exp(-jn\omega_0 t)\Big)$$

where $j^2 = -1$ and $\exp(u)$ denotes e^u, equation (1.1) becomes

$$f(t) = \frac{1}{2}a_0 + \frac{1}{2}\sum_{n=1}^{\infty}\{(a_n - jb_n)\exp(jn\omega_0 t) + (a_n + jb_n)\exp(-jn\omega_0 t)\}. \qquad (1.4)$$

If we define

$$c_0 = \frac{1}{2}a_0$$

$$c_n = \frac{1}{2}(a_n - jb_n), \quad n = 1, 2, 3, \ldots$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n), \quad n = 1, 2, 3, \ldots,$$

then equation (1.4) can be written in the *complex form*

$$f(t) = \sum_{n=-\infty}^{n=\infty} c_n \exp(jn\omega_0 t). \qquad (1.5)$$

where

$$c_n = \frac{\omega_0}{2\pi}\int_{-\pi/\omega_0}^{\pi/\omega_0} f(t)\exp(-jn\omega_0 t)\,dt. \qquad (1.6)$$

† For convenience in printing the notation exp( ) is frequently used for $e^{( )}$.