Theory of Reflection

of Electromagnetic and Particle Waves

by

John Lekner

Theory of Reflection

of Electromagnetic and Particle Waves

by

John Lekner

Department of Physics
Victoria University of Wellington
New Zealand

1987 **MARTINUS NIJHOFF PUBLISHERS**
a member of the KLUWER ACADEMIC PUBLISHERS GROUP
DORDRECHT / BOSTON / LANCASTER

Distributors

for the United States and Canada: Kluwer Academic Publishers, P.O. Box 358, Accord Station, Hingham, MA 02018-0358, USA
for the UK and Ireland: Kluwer Academic Publishers, MTP Press Limited, Falcon House, Queen Square, Lancaster LA1 1RN, UK
for all other countries: Kluwer Academic Publishers Group, Distribution Center, P.O. Box 322, 3300 AH Dordrecht, The Netherlands

Library of Congress Cataloging in Publication Data

Lekner, John.
Theory of reflection.

(Developments in electromagnetic theory and applications ; 3)
Includes indexes.
1. Reflection (Optics) 2. Electromagnetic waves--Transmission. I. Title. II. Title: Particle waves. III. Series.
QC425.L55 1987 530.1'24 86-24511

ISBN 90-247-3418-5 (this volume)
ISBN 90-247-2888-6 (series)

Copyright

PRINTED IN THE NETHERLANDS

Preface

This book is written for scientists and engineers whose work involves wave reflection or transmission. Most of the book is written in the language of electromagnetic theory, but, as the title suggests, many of the results can be applied to particle waves, specifically to those satisfying the Schrödinger equation. The mathematical connection between electromagnetic s (or TE) waves and quantum particle waves is established in Chapter 1. The main results for s waves are translated into quantum mechanical language in the Appendix. There is also a close analogy between acoustic waves and electromagnetic p (or TM) waves, as shown in Section 1-4. Thus the book, though primarily intended for those working in optics, microwaves and radio, will be of use to physicists, chemists and electrical engineers studying reflection and transmission of particles at potential barriers. The techniques developed here can also be used by those working in acoustics, oceanography and seismology.

Chapter 1 is recommended for all readers: it introduces reflection phenomena, defines the notation, and previews (in Section 1-6) the contents of the rest of the book. This preview will not be duplicated here. We note only that applied topics do appear: two examples are the important phenomenon of attenuated total reflection in Chapter 8, and the reflectivity of multilayer dielectric mirrors in Chapter 12. The subject matter is restricted to linear classical electrodynamics in non-magnetic media, and the corresponding particle analogues. Phenomena in non-linear and quantum optics are not covered. Even with these restrictions the book has grown larger than originally planned.

My interest in the theory of reflection was stimulated by David Beaglehole's studies of interfaces by polarization modulation ellipsometry, and work in this field was made more enjoyable by the many discussions we have had.

The editor of this series, John Heading, has kindly made many comments and suggestions as the book was being written. This generosity with his time and expertise is greatly appreciated, especially in view of his widespread commitments.

Almost all the book was written while I was a Visiting Fellow at the Department of Applied Mathematics of the Australian National University. The warm hospitality of Barry Ninham and his colleagues combined with the beauty of the Australian bush, coast and wildlife to make the year here a delight. My family only wished that more time could be given to exploring Australia, and less to the book!

I was fortunate to have overlapped with Colin Pask for part of the stay here. Through incisive comments he influenced the form and content of the book, especially the chapters on Riccati-type equations, inversion problems, and pulse and beam reflection.

Finally, special thanks are due to Kayleen Scott and Diana Wallace, who produced a professional word-processed text while coping with all the usual demands on their time.

JOHN LEKNER
Canberra, June 1986

Contents

Preface . . . . **XI**

Chapter 1. Introducing reflection **1**
1-1 The electromagnetic s wave 1
1-2 The electromagnetic p wave 5
1-3 Particle waves 10
1-4 Acoustic waves 13
1-5 Scattering and reflection 17
1-6 A look ahead 20

Chapter 2. Exact results **33**
2-1 Comparison identities, and conservation and reciprocity laws 33
2-2 General expressions for r_s and r_p 38
2-3 Reflection at grazing incidence, and the existence of a Brewster angle 42
2-4 Reflection by a uniform layer 44
2-5 Other exactly solvable profiles 50

Chapter 3. Reflection of long waves **61**
3-1 Integral equation and perturbation theory for the s wave 61
3-2 The s wave to second order in the interface thickness 64
3-3 Integral invariants 66
3-4 $|r_p|^2$ and r_p/r_s to second order 68
3-5 Reflection by a thin film between like media 71
3-6 Six profiles and their integral invariants 73

Chapter 4. Variational theory **77**
4-1 A variational expression for the reflection amplitude 77
4-2 Variational estimate for r_s in the long wave case 79
4-3 Exact, perturbation and variational results for the sech^2 profile. . . . 80
4-4 Variational theory for the p wave 83
4-5 Reflection by a non-uniform layer between like media 85
4-6 The Hulthén–Kohn variational method applied to reflection 89
4-7 Variational estimates in the short wave case 90

Chapter 5. Equations for the reflection amplitudes 93
5-1 A first order non-linear equation for an s wave reflection coefficient . 93
5-2 An example: reflection by the linear profile 95
5-3 Differential equation for a p wave reflection coefficient 97
5-4 Upper bounds on R_s and R_p 98
5-5 Long wave approximations 101
5-6 Differential equations for the reflection amplitudes 103
5-7 Weak reflection: the Rayleigh approximation 104
5-8 Iteration of the integral equation for r 106

Chapter 6. Reflection of short waves 109
6-1 Short wave limiting forms for some solvable profiles 109
6-2 Approximate waveforms 113
6-3 Profiles of finite extent with discontinuities in slope at the endpoints . 115
6-4 Reflection amplitude estimates from a comparison identity 118
6-5 Perturbation theory for short waves 121
6-6 Short wave results for r_p and r_p/r_s 124
6-7 A single turning point: total reflection 129
6-8 Two turning points, and tunnelling. 134

Chapter 7. Anisotropy 141
7-1 Anisotropy with azimuthal symmetry. 141
7-2 Ellipsometry off a thin film on an isotropic substrate 144
7-3 Thin film on an anisotropic substrate. 147
7-4 General results for anisotropic stratifications with azimuthal symmetry 148
7-5 Differential equations for the reflection amplitudes 149
7-6 Reflection from the ionosphere 151

Chapter 8. Absorption 155
8-1 Fresnel reflection formulae for an absorbing medium 156
8-2 General results for reflection by absorbing media 160
8-3 Dielectric layer on an absorbing substrate. 161
8-4 Absorbing film on a transparent substrate. 162
8-5 Thin non-uniform absorbing films 164
8-6 Attenuated total reflection; surface waves 168
8-7 Reflection by a diffuse absorbing interface: the tanh profile 176

Chapter 9. Inverse problems. 179
9-1 Reflection at a sharp boundary 180
9-2 Uniform film between like media 182
9-3 Synthesis of a profile from r as a function of wavenumber. 184

Chapter 10. Pulses, finite beams 191
10-1 Reflection of pulses: the time delay 191
10-2 Phase change on total internal reflection 194
10-3 Reflection of beams: the lateral beam shift 199

Chapter 11. Rough surfaces **205**
11-1 Reflection from rough surfaces: the Rayleigh criterion 205
11-2 Corrugated surfaces: diffraction gratings 206
11-3 Scattering of light by liquid surfaces. 211
11-4 The surface integral formulation of scattering by rough surfaces . . 215

Chapter 12. Matrix methods **221**
12-1 Matrices relating the coefficients of linearly independent solutions . . 221
12-2 Matrices relating fields and their derivatives 224
12-3 Periodically stratified media 228
12-4 Multilayer dielectric mirrors 230
12-5 Reflection of long waves. 234
12-6 Absorbing stratified media: some general results 236
12-7 High transparency of an absorbing film in a frustrated total reflection configuration 238

Chapter 13. Numerical methods **241**
13-1 Numerical methods based on the layer matrices. 242
13-2 Variable step size, profile truncation, total reflection and tunnelling, absorption, and calculation of wavefunctions 246

Appendix. Reflection of particle waves **249**
A-1 General results 249
A-2 Some exactly solvable profiles. 252
A-3 Perturbation and variational theories 257
A-4 Long waves, integral invariants 259
A-5 Riccati-type equations; the Rayleigh approximation 261
A-6 Reflection of short waves 262
A-7 Absorption, the optical potential 264
A-8 Inversion of a model reflection amplitude. 267
A-9 Reflection of wavepackets 268

Author index **273**

Subject index **277**

1

Introducing reflection

Electromagnetic, acoustic and particle waves all scatter, diffract and interfere. Reflection is the result of the constructive interference of many scattered or diffracted waves originating from scatterers in a stratified medium. This fundamental many-body approach is hard to apply (two illustrations are given in Section 1–5). Usually one replaces the collection of scatterers by an effective medium whose properties are represented, as far as wave propagation is concerned, by a function of position and frequency (or energy), such as the dielectric function ε in the electromagnetic case, or the effective potential V in the quantum particle case. Electromagnetic and particle waves then satisfy the same kind of linear partial differential equation, with ε and V playing similar roles.

In a medium with planar stratification the functions ε and V depend on only one spatial variable, and the partial differential equations then separate. Snell's Law is a direct consequence of this separability of the spatial dependence. The differential equations, and the elementary reflection properties which follow from them, are derived for electromagnetic, particle, and acoustic waves in the first four Sections. The many-body, constructive interference, aspect of reflection is outlined in Section 1-5. Finally, Section 1-6 previews some of the main results in Chapters 1 to 13.

1-1 The electromagnetic *s* wave

The reflection of a plane electromagnetic wave at a planar interface between two media is completely characterized when solutions for two mutually perpendicular polarizations are known. The polarizations conventionally chosen are: one with its electric vector perpendicular to the plane of incidence (labelled *s*, from the German *senkrecht*, perpendicular), and the other with its electric vector parallel to the plane of incidence (labelled *p*).

We consider monochromatic waves, of angular frequency ω. The reflection of a general electromagnetic wave (a pulse, for example) can be analyzed as that of a superposition of monochromatic waves. For a given ω the time dependence of all fields is carried in the factor $e^{-i\omega t}$. (This is the convention in quantum and solid state physics, and much of optics. In radio and electrical engineering the factor $e^{i\omega t}$ is often used. With the convention used here the dielectric function has positive imaginary part in the case of absorption.) We will consider only *non-magnetic* media in this book. The electrodynamic properties of a medium are then contained in the dielectric function $\varepsilon(\mathbf{r}, \omega)$ which is the ratio of the permitivity of the medium at position **r** and angular frequency ω to that of the vacuum. The wave equations follow from Maxwell's two curl equations relating the electric field E and the magnetic field **B**:

$$\nabla \times \mathbf{E} = i\omega\mathbf{B} \qquad \text{or} \qquad \nabla \times \mathbf{E} = i\frac{\omega}{c}\mathbf{B}, \tag{1}$$

$$\nabla \times \mathbf{B} = -i\varepsilon\frac{\omega}{c^2}\mathbf{E} \qquad \text{or} \qquad \nabla \times \mathbf{B} = -i\varepsilon\frac{\omega}{c}\mathbf{E}. \tag{2}$$

(The equations on the left are in SI units, those on the right in Gaussian units; the difference lies in the positioning of the speed of light c. In reflection studies, theory and experiment deal in dimensionless ratios, and the choice of units is irrelevant. Even the formal distinction disappears from equation (5) onward.)

For a planar interface lying in the xy plane, and an electromagnetic wave propagating in the x and z directions, the s wave has $\mathbf{E} = (0, E_y, 0)$ and (1) gives

$$-\frac{\partial E_y}{\partial z} = i\frac{\omega}{c}B_x, \qquad \frac{\partial E_y}{\partial x} = i\frac{\omega}{c}B_z, \tag{3}$$

and $B_y = 0$. The other curl equation gives

$$\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = -i\varepsilon\frac{\omega}{c}E_y. \tag{4}$$

On eliminating B_x and B_z from (3) and (4), we obtain a second order partial differential equation for E_y,

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} + \varepsilon\frac{\omega^2}{c^2}E_y = 0. \tag{5}$$

For planar stratifications the dielectric function depends on one spatial variable, z. The partial differential equation is then separable, with

$$E_y(x, z, t) = e^{i(Kx-\omega t)}E(z), \tag{6}$$

where $E(z)$ satisfies the ordinary differential equation

$$\frac{d^2E}{dz^2} + q^2E = 0, \qquad q^2 = \varepsilon\frac{\omega^2}{c^2} - K^2 = k^2 - K^2. \tag{7}$$

The meanings of k, K and q are evident from (5), (6) and (7): $k = \varepsilon^{1/2}\omega/c$ is the local value of the wavevector, $K = k_x$ is the component of the wavevector along the interface, and $q = k_z$ is the component of the wavevector normal to the interface. For a plane wave incident from medium 1 as shown in Figure 1-1,

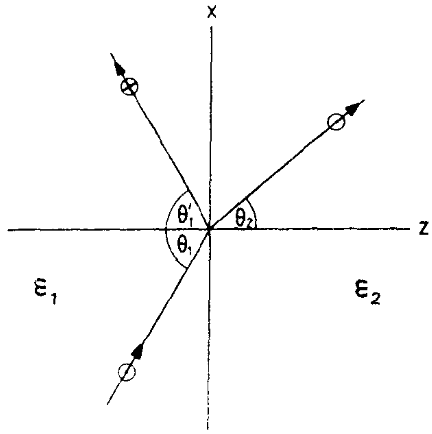

Figure 1-1. Reflection of the electromagnetic *s* wave at a planar interface between media characterized by dielectric constants ε_1 and ε_2. The figure is drawn for the air–water interface at optical frequencies, with $\varepsilon_1 \simeq 1$, $\varepsilon_2 \simeq (4/3)^2$.

the existence of the separation-of-variables constant $K(=k_{1x} = k'_{1x})$ implies

$$\varepsilon_1^{1/2} \sin\theta_1 = \varepsilon_1^{1/2} \sin\theta'_1 = \varepsilon_2^{1/2} \sin\theta_2, \tag{8}$$

where θ_1, θ'_1 and θ_2 are the angles of incidence, reflection, and transmission (or refraction).

Thus the fact that ε is a function of one spatial coordinate only, and the consequent separation of variables, implies the laws of reflection and refraction: the angle of reflection is equal to the angle of incidence, and the angles of incidence and refraction are related by Snell's Law. The refractive indices n_1 and n_2 of the two media, defined as coefficients in Snell's Law $n_1 \sin\theta_1 = n_2 \sin\theta_2$, are $\sqrt{\varepsilon_1}$ and $\sqrt{\varepsilon_2}$. Note that the laws of reflection-refraction do not depend on the transition between the two media being sharp: they are valid for an arbitrary variation of $\varepsilon(z)$ between the asymptotic values ε_1 and ε_2.

As ε attains its limiting values ε_1 and ε_2, $q = (\varepsilon\omega^2/c^2 - K^2)^{1/2}$ takes the limiting values

$$q_1 = \varepsilon_1^{1/2}\frac{\omega}{c}\cos\theta_1, \qquad q_2 = \varepsilon_2^{1/2}\frac{\omega}{c}\cos\theta_2. \tag{9}$$

(For $\theta_1 > \theta_c = \arcsin(n_2/n_1)$ there is total reflection, q_2 is imaginary, and θ_2 is complex. This is discussed along with the particle case in Section 1-3.) Snell's Law and the relationships between the wavevector components are incorporated together in Figure 1-2.

We now define the reflection and transmission amplitudes r_s and t_s in terms of the limiting forms of the solution of (7):

$$e^{iq_1z} + r_s\, e^{-iq_1z} \leftarrow E(z) \rightarrow t_s\, e^{iq_2z}. \tag{10}$$

The reflection amplitude is thus defined as the ratio of the coefficient of e^{-iq_1z} to that of e^{iq_1z}, the transmission amplitude as the coefficient of e^{iq_2z} when the incident wave

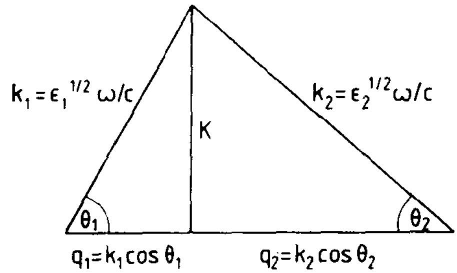

Figure 1-2. Graphical representation of $k_i^2 = q_i^2 + K^2$ and $K = k_1 \sin\theta_1 = k_2 \sin\theta_2$. The figure is drawn for the air–water interface, as in Figure 1-1.

e^{iq_1z} has unit amplitude. Theory aims to obtain general properties of the reflection and transmission amplitudes, and to develop methods for calculating these for a given dielectric function profile. The calculation is simple for the important step profile

$$\varepsilon_0(z) = \begin{cases} \varepsilon_1 & (z < 0) \\ \varepsilon_2 & (z > 0). \end{cases} \tag{11}$$

For this profile we obtain r_s and t_s from the continuity of E and dE/dz at $z = 0$. (If, for example, dE/dz were discontinuous, d^2E/dz^2 would have a delta function part, and (7) would not be satisfied.) For the step profile, E is given by the left and right sides of (10) for $z < 0$ and $z > 0$, respectively. The continuity of E and dE/dz at the origin gives

$$1 + r_{s0} = t_{s0}, \qquad iq_1(1 - r_{s0}) = iq_2t_{s0}. \tag{12}$$

Thus

$$r_{s0} = \frac{q_1 - q_2}{q_1 + q_2}, \qquad t_{s0} = \frac{2q_1}{q_1 + q_2}. \tag{13}$$

On using (8) and (9), the expressions (13) may be put into the Fresnel forms (Fresnel, 1823)

$$r_{s0} = \frac{\sin(\theta_2 - \theta_1)}{\sin(\theta_2 + \theta_1)}, \qquad t_{s0} = \frac{2\sin\theta_2\cos\theta_1}{\sin(\theta_2 + \theta_1)}. \tag{14}$$

The phases of the reflected and transmitted waves are specified only when the phase of the incident wave *and* the location of the interface are specified. The above equations are for the discontinuity in $\varepsilon(z)$ located at $z = 0$. In general, for the step located at z_1,

$$r_{s0} = e^{2iq_1z_1}\frac{q_1 - q_2}{q_1 + q_2}, \qquad t_{s0} = e^{i(q_1-q_2)z_1}\frac{2q_1}{q_1 + q_2}. \tag{15}$$

A special situation arises at grazing incidence ($\theta_1 \to \pi/2$, $q_1 \to 0$), when the incident and reflected waves are propagating in the same direction. Then the phase of the reflected wave *is* well-defined without specification of the interface location, and

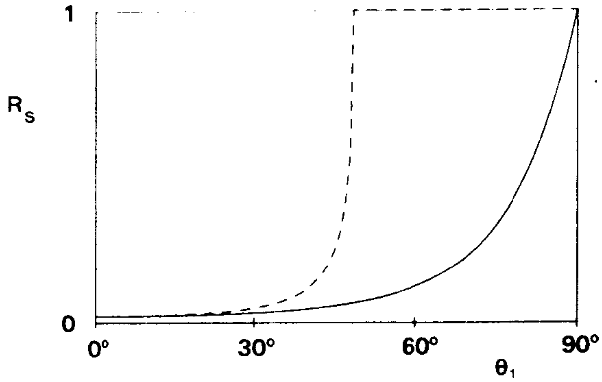

Figure 1-3. Step profile reflectivity for the s wave. The parameters are for the air–water interface at optical frequencies, as in Figures 1-1 and 1-2. The full curve is for light incident from air; the dashed curve for light incident from water shows total internal reflection for $\theta_1 > \theta_c \simeq \arcsin(3/4) \simeq 48.6°$.

$r_{s0} \to -1$ (even in the case of the total internal reflection, when q_2 is imaginary). The fact that $r_s \to -1$ at grazing incidence is a general property of all interfaces, as will be shown in Section 2-3.

The classical electromagnetic fields **E** and **B** are real quantities, and the complex notation is used for mathematical convenience. (Complex fields are intrinsic in the quantum theory of particles, however.) The physical reflected s wave is, for unit amplitude of the incident wave,

$$\mathrm{Re}\,\{r_s \exp i(Kx - q_1 z - \omega t)\} = \mathrm{Re}\,(r_s)\cos(Kx - q_1 z - \omega t) - \mathrm{Im}\,(r_s)\sin(Kx - q_1 z - \omega t).$$

The intensity is proportional to the time average of the square of this, namely

$$\tfrac{1}{2}[\mathrm{Re}\,(r_s)]^2 + \tfrac{1}{2}[\mathrm{Im}\,(r_s)]^2 = \tfrac{1}{2}|r_s|^2.$$

The incident intensity is proportional to the time average of $\cos^2(Kx + q_1 z - \omega t)$, which is 1/2. Thus, $R_s = |r_s|^2$ is the ratio of the reflected intensity to the incident intensity. This quantity is called the reflectivity, or reflectance. Figure 1-3 shows R_s for a sharp transition between air and water, with light incident from air, and from water.

1-2 The electromagnetic *p* wave

We again take the incident and reflected waves propagating in the zx plane, and the stratifications lying in xy planes. For the p wave, $\mathbf{B} = (0, B_y, 0)$; the Maxwell equation (1) gives

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = i\frac{\omega}{c}B_y, \tag{16}$$

while (2) implies $E_y = 0$ and

$$\frac{\partial B_y}{\partial z} = i\varepsilon \frac{\omega}{c} E_x, \qquad \frac{\partial B_y}{\partial x} = -i\varepsilon \frac{\omega}{c} E_z. \tag{17}$$

Elimination of E_x and E_z gives

$$\frac{\partial}{\partial x}\left(\frac{1}{\varepsilon}\frac{\partial B_y}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\varepsilon}\frac{\partial B_y}{\partial z}\right) + \frac{\omega^2}{c^2} B_y = 0. \tag{18}$$

When ε is a function of one spatial coordinate z, the laws of reflection and refraction again follow from the separability of (18). We set

$$B_y(x, z, t) = e^{i(Kx-\omega t)} B(z), \tag{19}$$

where K has the same meaning as for the s wave; then $B(z)$ satisfies the ordinary differential equation

$$\frac{d}{dz}\left(\frac{1}{\varepsilon}\frac{dB}{dz}\right) + \left(\frac{\omega^2}{c^2} - \frac{K^2}{\varepsilon}\right) B = 0. \tag{20}$$

When ε is constant (outside the interfacial region), the p wave equation has the same form as the s wave equation, with the same wavevector component q perpendicular to the interface. But within the interface there is an additional term proportional to the product of $d\varepsilon/dz$ and dB/dz. This term may be removed (and (20) converted to the form of the s wave equation (7)) in two ways. The first involves defining a new dependent variable

$$b = \left(\frac{\varepsilon_1}{\varepsilon}\right)^{1/2} B. \tag{21}$$

(The factor $\varepsilon_1^{1/2}$ makes identical the limiting forms of b and B in medium 1.) The equation satisfied by b is

$$\frac{d^2 b}{dz^2} + q_b^2 b = 0, \qquad q_b^2 = q^2 - \varepsilon^{1/2}\frac{d^2\varepsilon^{-1/2}}{dz^2} = q^2 + \frac{1}{2\varepsilon}\frac{d^2\varepsilon}{dz^2} - \frac{3}{4}\left(\frac{1}{\varepsilon}\frac{d\varepsilon}{dz}\right)^2. \tag{22}$$

This form of the p polarization equation is useful for special profiles, in particular the exponential profile, which has $\log\varepsilon$ linear in z, and the Rayleigh profile, which has $\varepsilon^{-1/2}$ linear in z. These are discussed in Chapter 2. It is also useful at short wavelengths, in the derivation of a perturbation theory for the p wave (Chapter 6).

The second transformation which removes the $(d\varepsilon/dz)\,(dB/dz)$ term is a dilation of the z variable in proportion to the local value of $\varepsilon(z)$: we define a new independent variable Z by

$$dZ = \varepsilon\, dz. \tag{23}$$

Then, as may be seen on division of (20) by ε, the p wave equation reads

$$\frac{d^2 B}{dZ^2} + Q^2 B = 0, \qquad Q^2 = \frac{1}{\varepsilon}\frac{\omega^2}{c^2} - \frac{K^2}{\varepsilon^2}. \tag{24}$$

This equation, in terms of the dilated z variable, and a reduced normal component of the wavevector, $Q = q/\varepsilon$, will be useful in many applications thoughout this book.

The p wave reflection and transmission amplitudes are defined in terms of the limiting forms of $B(z)$:

$$e^{iq_1z} - r_p e^{-iq_1z} \leftarrow B(z) \rightarrow \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^{1/2} t_p e^{iq_2z} \tag{25}$$

The reason for the factors -1 and $(\varepsilon_2/\varepsilon_1)^{1/2}$ multiplying r_p and t_p is that we wish r_s and r_p and t_s and t_p to refer to the same quantity, here chosen to be the electric field. (This is not the only convention in use: some authors have the opposite sign on r_p.) The electric field components for the p wave are found from (2), (19) and (25) to have the limiting forms

$$\varepsilon_1^{-1/2} \cos\theta_1 e^{i(Kx-\omega t)}(e^{iq_1z} + r_p e^{-iq_1z}) \leftarrow E_x \rightarrow \varepsilon_1^{-1/2} \cos\theta_2 t_p e^{i(Kx+q_2z-\omega t)}, \tag{26}$$

$$-\varepsilon_1^{-1/2} \sin\theta_1 e^{i(Kx-\omega t)}(e^{iq_1z} - r_p e^{-iq_1z}) \leftarrow E_z \rightarrow -\varepsilon_1^{-1/2} \sin\theta_2 t_p e^{i(Kx+q_2z-\omega t)}. \tag{27}$$

The x-component of the electric field (tangential to the interface) thus has the reflection amplitude r_p, while the z-component (normal to the interface) has reflection amplitude $-r_p$.

At normal incidence there is no physical difference between the s and p polarizations: both have electric and magnetic fields tangential to the interface. For our geometry, E_z is zero at normal incidence, and (1) implies $\partial E_x/\partial z = i(\omega/c)B_y$. Thus B, the solution of (20) and (25), must be proportional to dE/dz, where E is the solution of (7) and (10). On substituting dE/dz for B in (20) (with K set equal to zero) the left side becomes

$$\frac{d}{dz}\left\{\frac{1}{\varepsilon}\left(\frac{d^2E}{dz^2} + \varepsilon\frac{\omega^2}{c^2}E\right)\right\},$$

and this is zero, by (7). Thus (20) is satisfied by dE/dz at normal incidence. The proportionality of B and dE/dz at normal incidence, when applied to the limiting forms (10) and (25), gives the equality of r_p with r_s and of t_p with t_s. (Proportionality of B and dE/dz could be replaced by equality of B and $(c/i\omega)\, dE/dz$, but then (25) would have to be modified by the factor $\varepsilon_1^{1/2}$.)

At a discontinuity in the dielectric function, B and $dB/\varepsilon dz = dB/dZ$ are continuous (from (20) or (24)). For the step profile $\varepsilon_0(z)$ defined by (11), B is equal to

$$B_0(z) = \begin{cases} e^{iq_1z} - r_{p0} e^{-iq_1z} & (z < 0) \\ \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^{1/2} t_{p0} e^{iq_2z} & (z > 0). \end{cases} \tag{28}$$

The continuity of B and $dB/\varepsilon\, dz$ at the origin gives

$$1 - r_{p0} = \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^{1/2} t_{p0}. \tag{29}$$

$$iQ_1(1 + r_{p0}) = iQ_2\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^{1/2} t_{p0}, \tag{30}$$

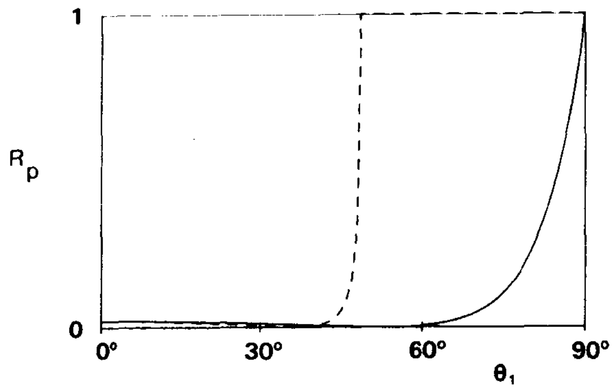

Figure 1-4. Step profile reflectivity for the p wave, for the air–water interface. The full curve is for light incident from air, the dashed curve for incidence from water. Note the zeros at the Brewster angles, arctan (4/3) $\simeq$ 53.1° and arctan (3/4) $\simeq$ 36.9°, respectively.

where $Q_1 = q_1/\varepsilon_1$ and $Q_2 = q_2/\varepsilon_2$. Thus (compare (13))

$$-r_{p0} = \frac{Q_1 - Q_2}{Q_1 + Q_2}, \qquad \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^{1/2} t_{p0} = \frac{2Q_1}{Q_1 + Q_2}. \tag{31}$$

On using (8) and (9) we obtain the Fresnel forms

$$r_{p0} = \frac{\tan(\theta_2 - \theta_1)}{\tan(\theta_2 + \theta_1)}, \qquad t_{p0} = \frac{2\sin\theta_2\cos\theta_1}{\sin(\theta_2 + \theta_1)\cos(\theta_2 - \theta_1)}. \tag{32}$$

The reflectivity of the p polarization off a discontinuity in the dielectric function is shown in Figure 1-4.

From (31) we see that the p wave shows zero reflection when $Q_1 = Q_2$, that is at the Brewster angle

$$\theta_B = \arctan\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^{1/2}. \tag{33}$$

It is apparent from (24) that this angle has special significance not only for a sharp transition between two media, but for diffuse profiles as well. This is because the wave equation in the dilated variable Z links two media with effective wavevector components Q_1 and Q_2, which are equal at this angle. The difference between the s and p effective wavevector components q and Q, and the reason for small p reflectivity at the Brewster angle, are illustrated in Figure 1-5. There we show q^2 versus z and Q^2 versus Z for the hyperbolic tangent profile

$$\varepsilon(z) = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2) - \tfrac{1}{2}(\varepsilon_1 - \varepsilon_2)\tanh z/2a, \tag{34}$$

for which

$$Z = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2)z - (\varepsilon_1 - \varepsilon_2)a \log\cosh(z/2a). \tag{35}$$

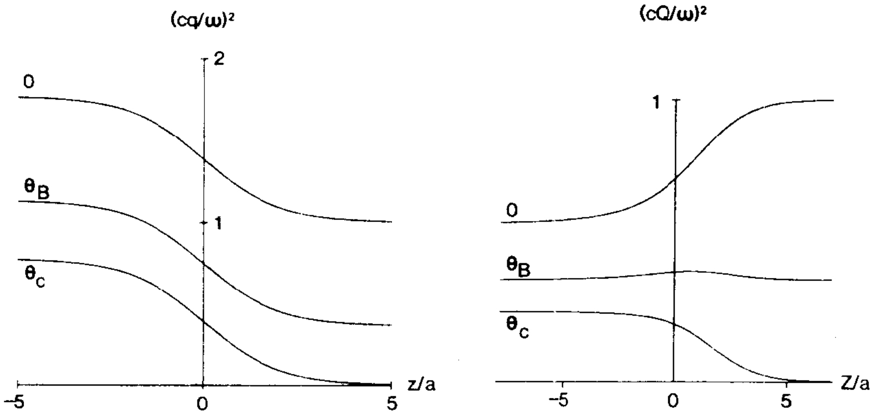

Figure 1-5. Squares of the normal wavevector component q and of the effective normal component Q for the s and p waves. The figure shows $q^2(z)$ and $Q^2(Z)$ for the hyperbolic tangent dielectric function profile, at three angles of incidence. The upper curve (in each case) is for normal incidence, the middle curve is at the Brewster angle $\theta_B = \arctan(\varepsilon_2/\varepsilon_1)^{1/2}$, and the lower curve is at the critical angle for total internal reflection, $\theta_c = \arcsin(\varepsilon_2/\varepsilon_1)^{1/2}$. The dielectric constants $\varepsilon_1 = (4/3)^2$ and $\varepsilon_2 = 1$ approximate the water–air interface. Water is on the left in both diagrams.

At the Brewster angle θ_B,

$$Q_1^2 = Q_2^2 = \frac{(\omega/c)^2}{\varepsilon_1 + \varepsilon_2} = Q_B^2, \tag{36}$$

$$K^2 = \varepsilon_1 \varepsilon_2 Q_B^2 = K_B^2. \tag{37}$$

From (24), a general profile $\varepsilon(z)$ has Q^2 at the Brewster angle given by

$$Q^2(\theta_B, z) = \frac{\omega^2}{c^2}\left\{\varepsilon(z) - \frac{\varepsilon_1\varepsilon_2}{\varepsilon_1 + \varepsilon_2}\right\}\Big/\varepsilon^2(z). \tag{38}$$

Thus the bump in Q^2 at the Brewster angle (see Figure 1-5) has the analytic form

$$Q^2(\theta_B, z) - Q_B^2 = \frac{\omega^2}{c^2}\frac{(\varepsilon_1 - \varepsilon)(\varepsilon - \varepsilon_2)}{\varepsilon^2(\varepsilon_1 + \varepsilon_2)}. \tag{39}$$

The p wave equation in the Z,Q notation has reflection at θ_B due to the small variation in the effective wavevector component Q as given by (39). For the step profile, ε is either ε_1 or ε_2, and there is no variation in Q and thus no reflection.

A common explanation for the small reflection of the p polarization at θ_B is in terms of the angular dependence of the dipole radiation from each atom or molecule which produces the transmitted and reflected waves. The far-field radiation pattern of a dipole has zero amplitude along the line of oscillation of the dipole (see Section 1-5, equation (78)); this is the reason for the polarization of light from the sky. We see from (32) that r_{p0} is zero when $\theta_1 + \theta_2 = \pi/2$, that is when the refracted and reflected waves are at a right angle (see Figure 1-6). The argument goes that at this angle of incidence there is no radiation from the accelerated electrons in the material to produce a p-polarized signal in the direction of specular