MATHEMATICAL MODELS OF TURBULENCE

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Lectures in

Mathematical Models of Turbulence

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Preface

This book has arisen directly from a short course of postgraduate lectures delivered by the authors at Imperial College during January 1971. The lectures were designed to convey the main concepts of turbulence modelling, and to show what progress had been made and what problems remained; the book has the same purpose.

The format of the book is also similar to that of the lectures. The latter proceeded by way of projected slides, accompanied by a connecting oral text. The book reproduces the majority of the slides, each with its portion of text, in a page-by-page arrangement corresponding to the order and duration of oral delivery. This was done for more than our own convenience as authors; we believe that the reader will also benefit from having topics brought squarely to his attention by the frame-like action of a single page.

We have made few amendments to the text of the lecture, apart from stylistic ones; and even here we have endeavoured to remain closer to the style of the lecture room than that of the textbook. This preference accords, we believe, with the nature of the subject, which is new, in rapid growth, and still experimental in respect of form. We have preferred to be suggestive and provocative rather than comprehensive and final; for the last two qualities are hardly attainable as yet.

Footnotes and references have been provided since the lectures were delivered; but relatively little recent material could be supplied without their swelling disproportionately.

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Lecture 1

Introduction

1.1 Preliminary remarks

When industrial historians assess the notable happenings of the 1970's they may record that this was a decade which saw a revolution in the design of engineering equipment. The agent of revolution is the digital computer with its ability to perform quickly and cheaply millions of arithmetic operations. Already, in problems of stress distributions in solids, designers turn to computer simulacra of structures for which formerly scale models were laboriously fabricated.

This pattern is now spreading to the design of process equipment where plant performance depends crucially upon transport and combustion processes through the fluids involved. Already, a few sections of the industry are turning to computer-based design-evolvement programmes; the practice seems certain to spread. We thus foresee that by 1980 computer experiments will largely have displaced experiments of the physical kind.

For a computer to provide realistic simulation of a flow process, it has to be supplied with a set of instructions (the computer program) which embodies the implications of the conservation laws of momentum, mass and energy, appropriate to a fluid in motion. Computer programs with these capabilities are now available for quite general two-dimensional flows and for certain three-dimensional ones as well. For laminar flows, predictions generated by these programs can, with due care, be as reliable as any experiment.

The same is not generally true when the flow is turbulent; for the laws governing the flux of momentum or heat through a turbulent field remain to be securely established. Nevertheless, in the past few years new approaches have brought much greater reliability to the calculation of turbulent flow phenomena. In this course of lectures, our purpose is to convey the essentials of these new practices.

1.2 The rationale of the turbulence-models approach

In principle, there is no need to adopt special practices for turbulent flow; for the Navier-Stokes equations apply equally to a turbulent motion as to a laminar one. All that is required, one might suppose, is a computer program to solve the equations.

But this is not a passable route at present. The reason is that important details of turbulence are small-scale in character; for example, eddies responsible for the decay of turbulence in a gaseous flow are typically about 0.1 mm. Now to solve the equations we must use a numerical procedure that calculates the value of variables at a number of discrete points in space. If this number were 10^5, which would stretch the storage capacity of any existing computer, we could still scarcely cover adequately one cubic centimetre of space.

Fortunately there is no need for an engineer to consider the details of turbulence; he is usually concerned only with its time-averaged effects, even when the mean flow is unsteady. Indeed, if we were given the time-dependent behaviour of a body of fluid, we should do nothing with the data but integrate them to extract time-averaged properties. This recognition affords us the following means of escape: we base predictions of turbulent flows on only the time-averaged properties of turbulence. Since these vary much more gradually in space, no excessively fine grid is needed.

The process of time-averaging, however, causes statistical correlations involving fluctuating velocities and temperatures to appear in the conservation equations. We have no direct way of knowing the magnitudes of these terms; we must therefore approximate or 'model' their effect in terms of quantities we can determine. Thus, by a 'model of turbulence' we mean a set of equations which, when solved with the mean-flow equations, allows calculation of the relevant correlations and so simulates the behaviour of real fluids in important respects.

1.3 Contents of the present lecture

- The requirements of a turbulence model
- Algebraic turbulent-viscosity models
- Differential turbulent-viscosity models
- Models employing differential equations for τ
- Comparison and summary

This, the first lecture of the course, will introduce the demands that turbulence models must meet; it enumerates the main classes of model to be considered, and makes a preliminary comparison and assessment of them.

As the box implies, there are three main types of model, the first two of which employ Boussinesq's (1877) suggestion that the stress~strain law for time-averaged turbulent flows could be represented in the same form as that for a Newtonian fluid in laminar motion. We shall distinguish those types of model in which the 'turbulent viscosity' is found by way of algebraic formulae, involving only properties of the mean-velocity profile as unknowns, from those in which it is determined from the solution of differential equations for one or more properties of the turbulent motion.

The third class of model to be considered includes those which (in the mean-momentum equations at any rate) dispense with the notion of effective turbulent transport properties and, instead, provide differential transport equations for the turbulent fluxes themselves.

1.4 Global or local accounting for influences of turbulent fluxes?

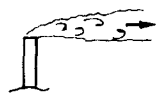
Research workers were, of course, suggesting ways of accounting for the effects of turbulence on the mean-flow behaviour long before it became practical to solve the partial-differential form of the convective transport equations. Thus, for boundary-layer flows, so-called integral procedures offered a powerful and comparatively flexible approach. But to calculate turbulent flows, these procedures needed to incorporate, implicitly or explicitly, suppositions about the global influence of turbulence on the mean-flow evolution. For example, Head (1960) supposed that the rate at which large-scale turbulent motions draw free-stream fluid into a boundary layer is determined solely by a shape parameter of the mean velocity profile.

However such global hypotheses are valid only over restricted ranges of conditions; accuracy and width of applicability cannot be achieved without solving the partial differential equations; and to solve these equations requires that local properties of turbulence be known. The present lectures, therefore, will be concerned exclusively with mathematical models of turbulence which provide local information.

1.5 The flow types of interest

Turbulence models are required for flow:

- remote from walls:



- adjacent to one wall:

- within ducts:

- with recirculation:

This box provides examples of the kinds of turbulent motion with which an engineer is commonly concerned. They are all turbulent 'shear flows' where because of mean velocity gradients, the turbulence is able to extract energy from the mean flow; the turbulent motion is thus generally self-sustaining.

As the sketches show, the flows in question include:- those which are remote from walls, such as the smoke plume from an industrial stack; flows near one wall, such as that in the vicinity of a turbine blade; and those developing within a duct. In some applications, the fluid motion possesses a single dominant direction with turbulent fluxes important only at right angles to this direction. We refer to these flows as of boundary-layer type; other things being equal, a simpler model of turbulence suffices than for flows where recirculatory motion is present.

Sometimes the flow is effectively two-dimensional (as in conical diffusers or through certain types of heat exchanger); in others (and these are in the majority) three-dimensional influences are of substantial importance. Again, it seems likely that the lower the level of dimensionality of the flow, the less complete need be our accounting of the turbulent motion.

1.6 Some physical processes affected by the turbulent motion

The local turbulence structure influences the rates of:

- transport of mean momentum.
- transport of heat and chemical species.
- decay of temperature fluctuations.
- chemical reaction.
- droplet evaporation. etc.

This box mentions some of the consequences of turbulent interaction that a model of turbulence may be called upon to predict. The existence of a correlation between the stream-wise and cross-stream components of fluctuating velocity gives rise to a lateral flux of streamwise momentum. This momentum flux may conveniently be interpreted as an additional shear stress arising from the turbulent motion; away from the vicinity of a wall, this exceeds by far the shear stress arising from molecular viscosity. Likewise the fluxes of enthalpy and chemical species are usually dominated by turbulence interactions.

It is not only the transport processes (of heat, momentum and matter) that are influenced by the presence of turbulence. Rates of decay of fluctuations, and rates of homogeneous chemical reaction and droplet vaporisation are equally affected; and these are often of great practical importance. All these interactions must therefore find a place in our turbulence models.

1.7 Some factors affecting the turbulent motion

- viscosity (e.g. the viscous sublayer)
- high Mach numbers
- combustion
- high swirl rates
- external force fields e.g. buoyancy, m.h.d. influences

In many circumstances, the local structure of the turbulence can be supposed to be adequately identified by two properties; these we can take to be the local velocity and the length scale of the turbulent motion. It is these circumstances to which the weight of our attention will be directed in this course.

However, within what is usually a very thin region near a wall, the scale and intensity of the turbulent motions are so diminished by the adjacent boundary that the effective Reynolds number of the eddies is quite small; small enough for laminar viscosity to exert direct influence on the turbulence. Likewise, when the Mach number of the flow exceeds about 5, it seems probable that the local structure is influenced by the appreciable density fluctuations in the flow. We also know that combustion or high rates of swirl may affect the local structure, as may external forces such as buoyancy or magnetohydrodynamic effects.

In principle, all these effects can be embraced within the framework of the turbulence models to be discussed in this course of lectures. Even though research has not proceeded far enough for quantitative certainty as yet, we can at least perceive the patterns into which the phenomena appear to fall.

1.8 Some desirable attributes of turbulence models

Perhaps we can best convey what is required of a turbulence model by naming four attributes we should like it to possess; they are:

- width of applicability,
- accuracy,
- economy of computer time,
- simplicity.

The designer wants a single set of equations simulating the turbulence action to serve him over the complete range of geometries and other parameters scanned in the search for the best design. He wants sufficient accuracy for the designed performance to differ negligibly from the actual; and the total economic expenditure, both in terms of manpower and of computing time, must be an acceptably small fraction of total investment.

Clearly, from the above, what constitutes the 'best' model of turbulence will differ according to the problem under consideration. Moreover, the more direct knowledge we have of the flow, the greater is the chance that a simple description of turbulence can be made to suffice. On one matter we can be quite definite, however: if sacrifice of simplicity and economy does not bring tangible benefits by way of greater accuracy and width of applicability, then the model may be referred back to its originator for further development.

1.9 The turbulent-viscosity concept

- Boussinesq (1877):

  replaced $\tau \equiv -\rho\,\overline{u'v'}$ by $\mu_t \dfrac{\partial u}{\partial y}$.

- μ_t is a property of the local state of the turbulence.
- How is μ_t determined?

Perhaps the first move towards a model of turbulence can be attributed to Boussinesq (1877). More than ninety years ago, he suggested that the effective turbulent shear stress, arising from the cross-correlation of fluctuating velocities, could be replaced by the product of the mean velocity gradient and a quantity termed the 'turbulent viscosity'.

Unlike μ, the molecular viscosity, μ_t is not a property of the fluid. Its value will vary from point to point in the flow, being largely determined by the structure of the turbulence at the point in question; at least, that is what we shall presume.

The introduction of μ_t provides a framework for constructing a turbulence model, but it does not itself constitute a model; for there remains the task of expressing the turbulent viscosity in terms of known or calculable quantities. The next few pages survey the ways in which different workers have accomplished this.

1.10 Some algebraic formulae for μ_t: 1. Prandtl's mixing-length hypothesis

- Proposal: $\mu_t = \rho\, \ell_m^{\ 2} \left|\frac{\partial u}{\partial y}\right|$.
- ℓ_m, the mixing length, must be prescribed.

  For boundary-layer flows, a few simple rules serve for its prescription.
- This model provides the subject for lecture 2.

Pre-eminent among the models which employ algebraic relations for μ_t is Prandtl's (1925) proposal, which has become known as the mixing-length hypothesis. For nearly-two-dimensional boundary-layer flows, particularly those developing over rigid boundaries, the mixing-length hypothesis combines a good mixture of the attributes named on page 8, so good indeed that only the best of the more comprehensive simulations of turbulence can surpass it.

The hypothesis is that the turbulent viscosity is equal to the local product of the density, of the magnitude of the mean rate of strain, and of the square of a characteristic length scale of the turbulent motion; this length scale we call the mixing length, ℓ_m. The mixing length must be prescribed algebraically; but, in boundary-layer flows, whether near to or remote from walls, a few simple rules usually serve for its prescription.

At Imperial College, we have been using the model extensively since 1966; and, more elegant approaches notwithstanding, there are many problems where this is the model most to be commended. For this reason we shall be devoting the next lecture to an elucidation of its application and performance.

1.11 Some algebraic formulae for μ_t: 2. von Kármán's similarity hypothesis

- Proposal: $\mu_t = \rho \, \ell_m^2 \left|\frac{\partial u}{\partial y}\right|$

  where $\ell_m \propto \left|\frac{\partial u}{\partial y} \bigg/ \frac{\partial^2 u}{\partial y^2}\right|$.

- The model circumvents the need to prescribe ℓ_m but fails to accord with experiment except for flows near walls.

An interesting contribution was made by von Kármán (1930). This, his so-called "similarity" hypothesis, may be expressed in a form akin to the mixing-length model. His analysis went further than Prandtl's, since it removed the necessity to prescribe the mixing-length profile. Von Kármán's analysis implied that ℓ_m was the ratio of the first to the second spatial derivatives of mean velocity.

Despite the ingenuity of von Kármán's proposal, his formula has not been extensively used. The reason is largely that the relationship for mixing length which his method predicts is not in agreement with measurements except in the vicinity of a wall; and the reason for that is, presumably, that the length scale is not determined solely by local properties of the mean flow, but is influenced by the properties at other locations in the vicinity.

One shortcoming of the von Kármán formula is immediately apparent when turbulent jets and mixing layers are considered. Their velocity profiles have inflexion points, where $\partial^2u/\partial y^2 = 0$, approximately at the position of maximum shear stress. Von Kármán's formula entails infinite mixing lengths there, and is therefore no help in computing the finite shear stress.

1.12 Some algebraic formulae for μ_t: 3. 'Eddy-viscosity' formulae

- Proposal: $\mu_t = \rho\, u_e y_e f\{y/\delta\}$

  u_e, y_e: characteristic global velocity and length scales.

  $f\{y/\delta\}$: prescribed function of (y/δ).
- Models of this type lack the width of applicability of the mixing-length hypothesis.

There is a further group of algebraic formulae for turbulent viscosity; their common feature is that the turbulent viscosity is supposed to be determined by velocity and length scales of the mean motion, u_e and y_e as indicated above. The term $f\{y/\delta\}$ refers to some function of position in the shear flow which possesses a limited degree of universality.

Various authors make different choices for u_e and y_e. In external wall boundary layers, the free-stream velocity and the displacement thickness of the boundary layer are usually adopted (Clauser, 1954, Mellor, 1963). In flow through a pipe, the so-called friction velocity and pipe radius are commonly chosen (Jonsson and Sparrow, 1966). The function $f\{y/\delta\}$ needs to be given a different form for each type of flow. So models of this type require the same kind of input as the mixing-length hypothesis.

Indeed they require more *ad hoc* adjustment than does the mixing-length hypothesis, when attention is shifted from one flow to another. For example, in a wall jet, which is a boundary layer where a jet of high-speed fluid is blown parallel to the surface, the boundary-layer displacement thickness is often negative; yet to predict the flow we do still need a positive μ_t. So clearly one must adopt a different interpretation of y_e; but it is a nuisance to have to do so and there seem to be no compensating advantages. For this reason we shall not be considering this type of model further.