MODELING AND ANALYSIS OF DYNAMIC SYSTEMS

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PREFACE

This book is intended for a first course in dynamic systems and is suitable for all engineering students regardless of discipline. It has been used at Rensselaer Polytechnic Institute for a one-semester course taken by every engineering student in his or her sophomore or junior year. While providing an exposure to dynamic systems for those whose interests lie in other areas, the material covered in this course serves as a basis for subsequent courses such as circuits and electronics, chemical process control, feedback systems, linear systems, vehicular dynamics and control, nuclear reactor control, biocontrol systems, systems physiology, and introduction to public systems. Because it covers such general topics as state variables, linearization of nonlinear models, numerical solution for the response, transfer functions, and feedback, this book can also be used for a general dynamic-systems course by students who have already completed a course in an area such as electrical circuits, machine dynamics, or chemical process dynamics. The rationale for the book is summarized in Section 1.1.

We assume that the reader has had differential and integral calculus and basic college physics, including mechanics and electrical phenomena. A course in differential equations should be taken at least concurrently. We have kept the mathematical level somewhere between the degree of rigor required for a mathematics book and the degree of expediency that raises inconsistencies for the discerning student and that often results in concepts that must be unlearned later. For example, the impulse has been treated in a manner that is consistent with distribution theory but that is no more difficult to grasp than the usual approach taken in introductory engineering books.

The organization of the book is indicated in the table of contents, and its scope and objectives are discussed in Section 1.5. Among the distinguishing features are the following:

1. A wide variety of physical systems is included, with one or more chapters on mechanical, electrical, electromechanical, thermal, and hydraulic systems. Each type is modeled in terms of its own fundamental laws and nomenclature.

2. The formulation of state-variable equations is included from the beginning, along with the more traditional input-output differential equation. We do not introduce matrix notation until the next-to-the-last chapter, however.

3. The technique for finding linearized models in terms of incremental variables is developed early in the book and used in a number of subsequent chapters. Incremental variables are also used to model linear systems with time-varying parameters.

4. The significance of the various components of a system's response—including the zero-input response, the zero-state response, and mode functions—is given more attention than is usual for a book at this level.

5. The numerical solution for the response of a system model is done in a way that does not emphasize a particular computer language. Numerical comparisons of the responses of nonlinear and linearized models are made, and problems requiring a programmable calculator or digital computer for their solution are included.

6. The use of Laplace transform techniques is deferred until after the introduction of time-domain and numerical solutions. The relationships of concepts such as the transfer function, poles and zeros, and frequency response to the time-domain solutions are emphasized.

7. The concluding chapter presents five case studies drawn from a variety of systems, including a sociological system.

We considered including discrete-time systems, but we did not do so because the length of the book would have become excessive. For this same reason, and because they are better suited for more advanced books, we do not discuss distributed and stochastic systems.

The majority of the material can be covered in a one-semester course, but the book can also be used as the basis for a year-long course. For schools on the quarter system, most of the material could be included in a two-quarter sequence. If necessary, any of Chapters 10, 11, or 14 to 18 can be omitted or abbreviated without loss of continuity.

In its various versions, the manuscript has been used at Rensselaer since 1973 by more than 2500 students. There are 130 examples and 420 problems. A solutions manual containing solutions to each of the problems is available.

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CHAPTER 1
INTRODUCTION

In this chapter we present the rationale for the book, define several terms that will be used throughout, and describe various types of systems. The chapter concludes with a description of the particular types of systems to be considered and a summary of the techniques that the reader should be able to apply after completing the book.

1.1 RATIONALE

The importance of understanding and being able to determine the dynamic response of physical systems has long been recognized. It has been traditional in engineering education to have separate courses in dynamic mechanical systems, circuit theory, chemical-process dynamics, and other areas. Such courses develop techniques of modeling, analysis, and design for the particular type of physical systems relevant to that specific discipline, even though many of the techniques taught in these courses have much in common. This approach tends to reinforce the student's view of such courses as isolated entities with little in common and to foster reluctance to apply what has been learned in one course to a new situation.

Another justification for considering a wide variety of different types of systems in an introductory book is that the majority of systems of practical interest contain components of more than one type. In the design of electronic circuits, for example, attention must be given to mechanical structure and to dissipation of the heat generated. Hydraulic motors and pneumatic process controllers are other examples of useful combinations of different types of elements. Furthermore, the techniques in this book can be applied not only to pneumatic, acoustical, and other traditional areas but also to systems that are quite different, such as sociological, physiological, economic, and transportation systems.

Because of the universal need for engineers to understand dynamic systems and because there is a common methodology applicable to such systems regardless of their physical origin, it makes sense to present them all together. This book considers both the problem of obtaining a mathematical description of a physical system and the various analysis techniques that are widely used.

1.2 ANALYSIS OF DYNAMIC SYSTEMS

Since the most frequent key word in the text is likely to be "system," it is appropriate to define it at the outset. A *system* is any collection of interacting elements for which there are cause-and-effect relationships among the variables. This definition is necessarily general, because it must encompass a broad range of systems. The important feature of the definition is that it tells us we must take interactions among the variables into account in system modeling and analysis, rather than treating individual elements separately.

Our study will be devoted to *dynamic systems*, for which the variables are time-dependent. In nearly all our examples, not only will the excitations and responses vary with time but at any instant the derivatives of one or more variables will depend on the values of the system variables at that instant. The system's response will normally depend on initial conditions, such as stored energy, in addition to any external excitations.

In the process of analyzing a system, two tasks must be performed: modeling the system and solving for the model's response. The combination of these steps is referred to as *system analysis*.

A *mathematical model*, or *model* for short, is a description of a system in terms of equations. The basis for constructing a model of a system is the physical laws (e.g., the conservation of energy and Newton's laws) that the system elements and their interconnections are known to obey.

The type of model sought will depend on both the objective of the engineer and the tools for analysis. If a pencil-and-paper analysis with parameters expressed in literal rather than numerical form is to be performed, a relatively simple model will be needed. To achieve this simplicity, the engineer should be prepared to neglect elements that do not play a dominant role in the system.

On the other hand, if a computer is available for carrying out simulations of specific cases with parameters expressed in numerical form, a comprehensive mathematical model that includes descriptions of both primary and secondary effects might be appropriate. The important notion is that a variety of mathematical models are possible for a system, and the engineer must be prepared to decide what form and complexity are most consistent with the objectives and the available resources.

The process of using the mathematical model to determine certain features of the system's cause-and-effect relationships is referred to as *solving the model*. For example, the responses to specific excitations may be desired for a range of parameter values, as guides in selecting design values for those parameters. As described in the discussion of modeling, this phase may include the analytical solution of simple models and the computer solution of more complex ones.

The type of equation involved in the model has a strong influence on the extent to which analytical methods can be used. For example, nonlinear differential equations can seldom be solved in closed form, and the solution of partial differential equations is far more laborious than that of ordinary differential equations. Computers can be used to generate the responses to specific numerical cases for complex models. However, the use of a computer for the solution of a complex model is not without its limitations. Models used for computer studies should be chosen with the approximations encountered in numerical integration in mind and should be relatively insensitive to system parameters whose values are uncertain or subject to change. Furthermore, it may be difficult to generalize results based only on computer solutions that must be run for specific parameter values, excitations, and initial conditions.

The engineer must not forget that the model being analyzed is only an approximate mathematical description of the system and is not the physical system itself. Conclusions based on equations that required a variety of assumptions and simplifications in their development may or may not apply to the actual system. Unfortunately, the more faithful a model is in describing the actual system, the more difficult it is to obtain general results.

One procedure is to use a simple model for analytical results and design and then to use a different model to verify the design by means of computer simulation. In very complex systems, it may be feasible to incorporate actual hardware components into the simulation as they become available, thereby eliminating the corresponding parts of the mathematical model.

1.3 CLASSIFICATION OF VARIABLES

A system is often represented by a box (traditionally called a "black box"), as shown in Figure 1.1. The system may have several *inputs*, or *excitations*, each of which is a function of time. Typical inputs are a force applied to a mass, a voltage source applied to an electrical circuit, and a heat source applied to a vessel filled with a liquid. In general discussions that are not related to specific systems, we shall use the symbols $u_1(t), u_2(t), \ldots, u_m(t)$ to denote the m inputs, shown by the arrows directed into the box.

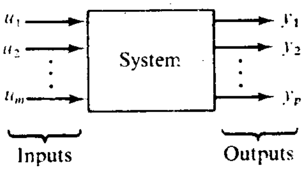

Figure 1.1 Black-box representation of a system.

Outputs are variables that are to be calculated or measured. Typical outputs are the velocity of a mass, the voltage across a resistor, and the rate at which a liquid flows through a pipe. The p outputs are represented in Figure 1.1 by the arrows pointing away from the box representing the system. They are denoted by the symbols $y_1(t), y_2(t), \ldots, y_p(t)$. There is a cause-and-effect relationship between the outputs and inputs. To calculate any one of the outputs for all $t \geq t_0$, we must know the inputs for $t \geq t_0$ and also the accumulated effect of any previous inputs. One approach to constructing a mathematical model is to find equations that relate the outputs directly to the inputs by eliminating all the other variables that are internal to the system. If we are interested only in the input-output relationships, eliminating extraneous variables may seem appealing. However, potentially important aspects of the system's behavior may be lost by deleting information from the model.

Another modeling technique is to introduce a set of *state variables*, which generally differs from the set of outputs but which may include one or more of them. The state variables must be chosen so that a knowledge of their values at any reference time t_0 and a knowledge of the inputs for all $t \geq t_0$ is sufficient to determine the outputs and state variables for all $t \geq t_0$. An additional requirement is that the state variables must be independent, meaning that it must not be possible to express one state variable as an algebraic function of the others. This approach is particularly convenient for working with multi-input, multioutput systems and for obtaining computer solutions. In Figure 1.2 the representation of the system has been modified to include the state variables denoted by the symbols $q_1(t), q_2(t), \ldots, q_n(t)$ within the box. The state variables can account for the important aspects of the system's behavior, regardless of the choice of output variables. Equations for the outputs can then be written as algebraic functions of the state variables, inputs, and time.

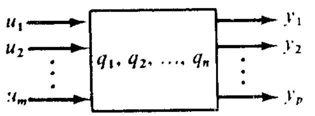

Figure 1.2 General system representation showing inputs, state variables, and outputs.

Whenever it is appropriate to indicate units for the variables and parameters, we shall use the International System of Units (abbreviated SI, from the French *Systeme International d'Unités*). A list of the units used in this book appears in Appendix A.

1.4 CLASSIFICATION OF SYSTEMS

Systems are grouped according to the types of equations that are used in their mathematical models. Examples are partial differential equations with time-varying coefficients, ordinary differential equations with constant coefficients, and difference equations. In this section we define and briefly discuss ways of classifying the models, and in the next section we indicate those categories that will be treated in this book. The classifications that we use are listed in Table 1.1.

SPATIAL CHARACTERISTICS

A *distributed system* does not have a finite number of points at which state variables can be defined. In contrast, a *lumped system* can be described by a finite number of state variables.

Table 1.1 Criteria for classifying systems

Criterion	Classification
Spatial characteristics	Lumped Distributed
Continuity of the time variable	Continuous Discrete-time Hybrid
Quantization of the dependent variable	Nonquantized Quantized
Parameter variation	Fixed Time-varying
Superposition property	Linear Nonlinear

To illustrate these two types of systems, consider the flexible shaft shown in Figure 1.3(a) with one end embedded in a wall and with a torque applied to the other end. The angle through which a point on the surface of the shaft is twisted depends on both its distance from the wall and the applied torque. Hence the shaft is inherently distributed and would be modeled by a partial differential equation. However, if we are only interested in the angle of twist at the right end of the shaft, we may account for the flexibility of the shaft by a rotational spring constant K and represent the effect of the distributed mass by the single moment of inertia J. Making these approximations results in the lumped system shown in Figure 1.3(b), which has the important property that its model is an ordinary differential equation. Because ordinary differential equations are far easier to solve than partial differential equations, converting from a distributed system to a lumped approximation is often essential if the resulting model is to be solved with the resources available.

Another example of a distributed system is an inductor that consists of a wire wound around a core, as shown in Figure 1.4(a). If an electrical

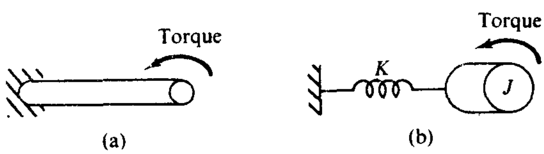

Figure 1.3 A torsional shaft and its lumped approximation.

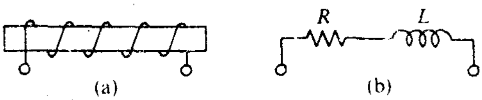

Figure 1.4 An inductor and its lumped approximation.

excitation is applied across the terminals of the coil, then different values of voltage would exist at all points along the coil, characteristic of a distributed system. To develop a lumped circuit whose behavior as calculated at the terminals closely approximates that of the distributed device, we might account for the resistance of the wire by a lumped resistance R and for the inductive effect related to the magnetic field by a single inductance L. The resulting lumped circuit is shown in Figure 1.4(b). Note that in these two examples (although not in all cases), the two elements in the lumped model do not correspond to separate physical parts of the actual system. The stiffness and moment of inertia of the flexible shaft cannot be separated into two physical pieces, nor can the resistance and inductance of the coil.

CONTINUITY OF THE TIME VARIABLE

A second basis for classifying dynamic systems is the independent-variable time. A *continuous system* is one for which the inputs, state variables, and outputs are defined over some continuous range of time (although the signals may have discontinuities in their waveshapes and not be continuous functions in the mathematical sense). A *discrete-time system* has variables that are determined at distinct instants of time and that are either not defined or not of interest between those instants. Continuous systems are described by differential equations and discrete-time systems by difference equations.

Examples of the variables associated with continuous and discrete-time systems are shown in Figure 1.5. In fact, the discrete-time variable $f_2(kT)$

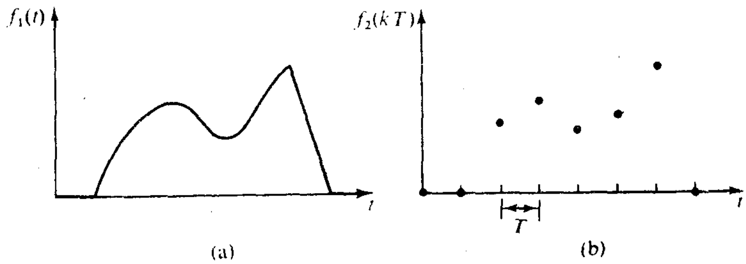

Figure 1.5 Sample variables. (a) Continuous. (b) Discrete-time.