

HANDBOOK OF MATHEMATICAL TABLES AND FORMULAS

FIFTH EDITION

RICHARD STEVENS BURINGTON, Ph.D.

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**HANDBOOK OF
MATHEMATICAL TABLES
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PREFACE

The excellent reception and wide use of the earlier editions of this book coupled with the valuable suggestions from its many users for additions have resulted in the publisher's requesting a revision and enlargement of the fourth edition.

As with the earlier editions, this book has been constructed to meet the needs of students and workers in mathematics, engineering, physics, chemistry, science, and other fields in which mathematical reasoning, processes, and computations are required.

The content and manner in which mathematical subjects are taught and used are constantly undergoing changes. This is true at all levels of instruction and use, from the elementary and secondary schools to colleges, institutes, and universities, as well as the many industries and laboratories where mathematics is so widely used. The applications of mathematics to many diverse fields continue to grow at an accelerated pace.

The large-scale use of various types of computing devices has greatly widened the use of certain kinds of mathematical methods and has had a marked impact on the teaching and utilization of mathematics. However, there are many aspects of the tradition, language, use, and teaching of mathematics which remain much the same as they have been for many years. The degree of acceptance and use of contemporary changes vary widely. The existence of many high-speed computers has not reduced the need for compilations of a mathematical character, or for resumes of mathematical methods, but rather it has increased the need for a variety of types of compilations, of which this book covers certain widely used areas.

These and other matters of concern have been carefully weighed by the author in preparing this edition. The present edition should meet the needs of both the contemporary and the more traditional users of mathematics.

The general spirit and arrangement of the fourth edition has been

retained. The elementary character of the book has been preserved. The size remains modest.

The first part of the book includes a summary of the more important formulas and theorems of algebra, trigonometry, analytical geometry, calculus, and vector analysis. The sections on sets, logic, relations, functions, algebraic structures, Boolean algebra, matrices, number systems, and elementary statistics have been retained. Comprehensive tables of derivatives and integrals are included. New sections on annuities, linear algebra, linear vector spaces, numerical analysis, solutions of linear and nonlinear systems of equations, approximate differentiation and integration, interpolation and approximation, calculus of finite differences, partial differential equations, Legendre and Bessel functions, Fourier analysis, Laplace transforms, and complex variable theory have been added. Tables of finite Fourier cosine and sine functions, and Laplace transforms have been included.

The second part of the book contains logarithmic and trigonometric tables to five places; tables of natural logarithms and exponential and hyperbolic functions; tables of squares, cubes, square roots, and cube roots; reciprocals and other numerical quantities; normal probability distribution functions; χ^2 distribution. The annuity tables have been enlarged to cover higher interest rates. A table of the summed Poisson distribution function has been added.

This edition is designed to serve as a companion to the *Handbook of Probability and Statistics with Tables*, second edition, by Richard S. Burington and Donald C. May, McGraw-Hill Book Company.

The entire book has been gone over, and numerous changes and additions have been made within the text as well as in the tables themselves. Much effort has been made to arrange the tables in such a manner that the user can and will interpret them properly, and with ease. Every effort has been taken to ensure accuracy.

The author wishes to acknowledge the valuable suggestions made by many users of the book, and to express his appreciation to his daughter Artha Jean Snyder and son-in-law Grant Snyder for their help and critical suggestions.

The author is greatly indebted to his wife, Jennet Mae Burington, for her continued interest and major assistance in the preparation of the manuscript and in the reading of the proof.

RICHARD STEVENS BURINGTON

How to Use the Table Locator

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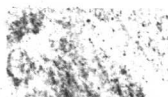


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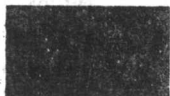
LOGARITHMS



**TRIGONOMETRIC
FUNCTIONS**



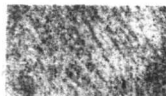
**EXPONENTIAL AND
HYPERBOLIC
FUNCTIONS**



**POWERS, ROOTS,
RECIPROCALs,
AREAS, AND
CIRCUMFERENCES**



**STATISTICS AND
PROBABILITY**



**INTEREST AND
ACTUARIAL
EXPERIENCE**



**CONSTANTS,
EQUIVALENTS,
AND CONVERSION
FACTORS**



**FOUR-PLACE
TABLES**



THIS BOOK CONTAINS

Part One: Summaries of formulas and theorems of

ALGEBRA	MATRICES
ANNUITIES AND INTEREST	LINEAR ALGEBRA
ELEMENTARY GEOMETRY	LINEAR VECTOR SPACES
TRIGONOMETRY	NUMERICAL ANALYSIS
ANALYTIC GEOMETRY	SOLUTION OF LINEAR
DIFFERENTIAL CALCULUS	SYSTEMS OF EQUATIONS
TABLE OF DERIVATIVES	SOLUTION OF NONLINEAR
TABLE OF SERIES	EQUATIONS
INTEGRAL CALCULUS	INTERPOLATION AND APPROXIMATION
TABLE OF INTEGRALS	NUMERICAL DIFFERENTIATION
VECTOR ANALYSIS	NUMERICAL INTEGRATION
SETS, RELATIONS, AND	FINITE DIFFERENCE CALCULUS
FUNCTIONS	DIFFERENTIAL EQUATIONS
LOGIC	LEGENDRE POLYNOMIALS
CALCULUS OF PROPOSITIONS	BESSEL FUNCTIONS
CALCULUS OF RELATIONS	PARTIAL DIFFERENTIAL EQUATIONS
ALGEBRAIC STRUCTURES	FOURIER SERIES AND TRANSFORMS
MATHEMATICAL SYSTEMS	TABLE OF FINITE SINE
GROUPS	AND COSINE TRANSFORMS
POSTULATES FOR COMMON	LAPLACE TRANSFORMS
STRUCTURES	TABLE OF LAPLACE TRANSFORMS
BOOLEAN ALGEBRAS	COMPLEX VARIABLE THEORY
NUMBER SYSTEMS	STATISTICS

Part Two: Tables of

LOGARITHMS	STATISTICS AND PROBABILITY
TRIGONOMETRIC FUNCTIONS	INTEREST AND ACTUARIAL
EXPONENTIAL AND	EXPERIENCE
HYPERBOLIC FUNCTIONS	CONSTANTS, EQUIVALENTS,
POWERS, ROOTS, RECIPROCALs,	AND CONVERSION FACTORS
AREAS, AND CIRCUMFERENCES	FOUR-PLACE TABLES

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PART ONE

*Formulas, definitions, and
theorems from elementary
mathematics*

1. ALGEBRA

1. FUNDAMENTAL LAWS

(a) Commutative law: $a + b = b + a$ $ab = ba$

(b) Associative law: $a + (b + c) = (a + b) + c$,
 $a(bc) = (ab)c$

(c) Distributive law: $a(b + c) = ab + ac$

2. LAWS OF EXPONENTS

$$\begin{aligned}a^x a^y &= a^{x+y} & (ab)^x &= a^x b^x & (a^x)^y &= a^{xy} \\a^0 &= 1 \quad \text{if } a \neq 0 & a^{-x} &= \frac{1}{a^x} & \frac{a^x}{a^y} &= a^{x-y} \\a^{x/y} &= \sqrt[y]{a^x} & a^{1/y} &= \sqrt[y]{a}\end{aligned}$$

3. OPERATIONS WITH ZERO

$$a - a = 0 \qquad a \times 0 = 0 \times a = 0$$

If $a \neq 0$, $\frac{0}{a} = 0$, $0^a = 0$, $a^0 = 1$ (Division by zero undefined)

4. COMPLEX NUMBERS (a number of the form $a + bi$ where a and b are real)

$$i = \sqrt{-1}, i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i, \text{ etc.}$$

$$a + bi = c + di \text{ if and only if } a = c, b = d$$

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i$$

5. LAWS OF LOGARITHMS (see explanation of Table 1).

If M, N, b are positive numbers and $b \neq 1$:

$$\log_b MN = \log_b M + \log_b N \qquad \log_b \frac{M}{N} = \log_b M - \log_b N$$

4 Algebra

$$\log_b M^p = p \log_b M \quad \log_b \sqrt[q]{M} = \frac{1}{q} \log_b M$$

$$\log_b \frac{1}{M} = -\log_b M \quad \log_b b = 1 \quad \log_b 1 = 0$$

Change of base of logarithms ($c \neq 1$):

$$\log_b M = \log_c M \log_b c = \frac{\log_c M}{\log_c b}$$

6. BINOMIAL THEOREM (n a positive integer)

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 \\ + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \dots + nab^{n-1} + b^n$$

where $n! = \underline{n} = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1)n$

7. EXPANSIONS AND FACTORS

$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$

$$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$$

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$$

$$a^2 - b^2 = (a - b)(a + b)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \dots + b^{n-1})$$

$$a^n - b^n = (a + b)(a^{n-1} - a^{n-2}b + \dots - b^{n-1})$$

for n an even integer

$$a^n + b^n = (a + b)(a^{n-1} - a^{n-2}b + \dots + b^{n-1})$$

for n an odd integer

$$a^4 + a^2b^2 + b^4 = (a^2 + ab + b^2)(a^2 - ab + b^2)$$

8. RATIO AND PROPORTION

If $a:b = c:d$ or $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$, $\frac{a}{c} = \frac{b}{d}$

If
$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \dots = k,$$

then
$$k = \frac{a + c + e + \dots}{b + d + f + \dots} = \frac{pa + qc + re + \dots}{pb + qd + rf + \dots}$$

9. CONSTANT FACTOR OF PROPORTIONALITY (OR VARIATION), k

If y varies directly as x , or y is proportional to x ,

$$y = kx$$

If y varies inversely as x , or y is inversely proportional to x ,

$$y = \frac{k}{x}$$

If y varies jointly as x and z ,

$$y = kxz$$

If y varies directly as x and inversely as z ,

$$y = \frac{kx}{z}$$

10. ARITHMETIC PROGRESSION

$$a \quad a + d \quad a + 2d \quad a + 3d \quad \dots$$

If a is the first term, d the common difference, n the number of terms, l the last term, and s the sum of n terms,

$$l = a + (n - 1)d \quad s = \frac{n}{2}(a + l)$$

The *arithmetic mean* of a and b is $(a + b)/2$.

11. GEOMETRIC PROGRESSION

$$a, ar, ar^2, ar^3 \dots$$

If a is the first term, r the common ratio, n the number of terms, l the last term, and S_n the sum of n terms,

$$l = ar^{n-1} \quad S_n = a \frac{r^n - 1}{r - 1} = \frac{rl - a}{r - 1}$$

If $r^2 < 1$, S_n approaches the limit S_∞ as n increases without limit,

$$S_\infty = \frac{a}{1-r}$$

The *geometric mean* of a and b is \sqrt{ab} .

12. HARMONIC PROGRESSION A sequence of numbers whose reciprocals form an arithmetic progression is called an *harmonic progression*. Thus

$$\frac{1}{a}, \quad \frac{1}{a+d}, \quad \frac{1}{a+2d}, \quad \dots$$

The *harmonic mean* of a and b is $2ab/(a+b)$.

13. PERMUTATION Each different arrangement of all or a part of a set of things is called a *permutation*. The number of permutations of n different things taken r at a time is

$$P(n, r) = {}_nP_r = n(n-1)(n-2) \cdots (n-r+1) = \frac{n!}{(n-r)!}$$

where

$$n! = n(n-1)(n-2) \cdots (1)$$

14. COMBINATIONS Each of the groups or relations which can be made by taking part or all of a set of things, without regard to the arrangement of the things in a group, is called a *combination*. The number of combinations of n different things taken r at a time is

$$\begin{aligned} C(n, r) = {}_nC_r &= \binom{n}{r} = \frac{{}_nP_r}{r!} \\ &= \frac{n(n-1) \cdots (n-r+1)}{r(r-1) \cdots 1} = \frac{n!}{r!(n-r)!} \end{aligned}$$

15. PROBABILITY If an event may occur in p ways and may fail in q ways, all ways being equally likely, the *probability* of its occurrence is $p/(p+q)$ and that of its failure to occur is $q/(p+q)$. (For further details see Richard S. Burington and Donald C. May, *Handbook of Probability and Statistics with Tables*, 2d ed., McGraw-Hill Book Company, New York; 1970.) (See Secs. 418 to 424.)

16. REMAINDER THEOREM (see Sec. 30) If the polynomial $f(x)$ is