Intermediate Mathematics of Electromagnetics

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Preface

This book introduces much of the mathematics relevant to electromagnetics and then applies that mathematics to some typical problems. Orthogonal functions, Green's functions, and Fourier transforms are presented in the first three chapters. The equations of electromagnetics in simple and non-simple media are introduced in Chapter 4. The normal modes and Green's functions for Laplace's and Helmholtz's equations in cartesian, cylindrical, and spherical coordinate systems are presented in Chapter 5. Finally, wave propagation in simple and non-simple media in unbounded and bounded space is treated in the last two chapters.

The problems are an integral part of the text and expand upon the material in each chapter. The answers to all of the problems are given so that the student can determine immediately the correctness of a solution. In addition, many of the problems in the first part of the book are chosen so that they later appear as parts of the solutions to more complicated problems in the latter part of the book.

The material in this text was developed at the University of Arizona and was originally used to supplement several of the graduate courses in electromagnetics. However, the material is suitable for a beginning graduate course in electromagnetics as well as a specialized course for seniors. It was presented to both seniors and beginning graduate students in electrical engineering at the

University of Texas at Arlington and was found to be satisfactory except that the pace was considerably slower with the seniors. The book is also suitable for independent study by students as well as being useful as a reference or supplement with courses in electromagnetics and applied mathematics.

In conclusion, the author wishes to express his appreciation to Dr. Andrew E. Salis of the University of Texas at Arlington for the opportunity to use this material in several courses.

D.C.S.

Contents

1

Orthogonal Functions and Introductory Mathematics

1.1 Introduction to Orthogonal Functions and Fourier Series

The concept of orthogonal functions is necessary for the solution of most problems in applied physics and engineering. These functions appear as solutions of the Sturm-Liouville equation which is one of the most basic equations in engineering. This equation will be discussed later after a few introductory remarks are made about Fourier series. The concept involved in the orthogonality of functions is but an extension of the concept of the orthogonality of vectors in n-dimensional space. For instance, let us define a vector

$$\mathbf{f} = \sum_{i=1}^{n} x_i \mathbf{a}_i$$

where $\mathbf{a}_i$ is the usual unit vector in the i-direction and x_i is the amplitude of the vector component of $\mathbf{f}$ in the i-direction. In three-dimensional space, n is 3, etc. If one wishes to evaluate x_j, this can be accomplished quite readily because of the orthogonality of the chosen coordinate system with the concomitant orthogonality of the unit vectors. Thus,

$$\mathbf{f} \cdot \mathbf{a}_j = \sum_{i=1}^{n} x_i \mathbf{a}_i \cdot \mathbf{a}_j = \sum_{i=1}^{n} x_i \delta_{ij} = x_j, \qquad \delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$$

where δ_{ij} is Kronecker's delta. This example illustrates the simplicity with which we can evaluate the amplitude of a particular vector component. One other item we shall need later is the norm N of the vector **f**. A knowledge of the value of N allows us to normalize **f**, i.e., find the unit vector $\mathbf{a}_f$ that is in the direction of **f**. As we already know

$$N = \mathbf{f} \cdot \mathbf{f} = \sum_{i=1}^{n} x_i \mathbf{a}_i \cdot \sum_{j=1}^{n} x_j \mathbf{a}_j = \sum_{\substack{i=1 \\ j=1}}^{n} x_i x_j \delta_{ij} = \sum_{i=1}^{n} x_i^2$$

so that we can evaluate the desired unit vector as

$$\mathbf{a}_f = \frac{\mathbf{f}}{\sqrt{N}}$$

All of the aforementioned ideas involving vectors are applicable to functions except that one of the summations is replaced by an integral. Thus, if we have a function $f(x)$ specified in the interval (a, b), we can expand it in a set of orthogonal functions $u_i(x)$ as follows:

$$f(x) = \sum_{i=1}^{\infty} c_i u_i(x) \tag{1-1}$$

The c_i are analogous to the x_i and the $u_i(x)$ are analogous to the $\mathbf{a}_i$. However, the summation now extends over an infinite number of integers and the orthogonality of the $u_i(x)$ is expressed as follows:

$$N\delta_{ij} = \int_a^b u_i(x) u_j(x)\, dx \tag{1-2}$$

where N is the norm of $u_i(x)$. Thus, we evaluate c_j in a very similar manner to that used to evaluate x_j:

$$\int_a^b f(x) u_j(x)\, dx = \int_a^b \sum_{i=1}^{\infty} c_i u_i(x) u_j(x)\, dx = c_j N$$

and letting $i \longrightarrow j$,

$$c_i = \frac{1}{N} \int_a^b f(x) u_i(x)\, dx \tag{1-3}$$

We shall see later that the integrand in (1-2) should include a weighting function $w(x)$. We have replaced $w(x)$ by unity here in order to avoid unnecessary complications.

Let us now proceed to Fourier's problem[1] as a specific example of these introductory concepts. His problem was the approximation of a function

[1] A. Sommerfeld, *Partial Differential Equations in Physics*, Academic Press, Inc., New York, 1949, p. 2.

$f(x)$ specified in the interval $(-\pi, \pi)$ by a sum of $2n + 1$ sinusoidal terms as follows:

$$S_n(x) = \sum_{k=0}^{n} A_k \cos kx + \sum_{k=1}^{n} B_k \sin kx$$

The question that had to be resolved was how to choose the unknown coefficients A_k and B_k. The sinusoids $\cos kx$ and $\sin kx$ are the orthogonal functions here and satisfy the relations

$$\int_{-\pi}^{\pi} \cos kx \cos \ell x \, dx = \frac{2\pi\delta_{k\ell}}{\epsilon_k}, \qquad \epsilon_k = \begin{cases} 1, & k = 0 \\ 2, & k > 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin kx \sin \ell x \, dx = \pi\delta_{k\ell}$$

$$\int_{-\pi}^{\pi} \sin kx \cos \ell x \, dx = 0$$

where ϵ_k is Neumann's number. To answer this question, Fourier defined an error term

$$\Delta_n(x) = f(x) - S_n(x)$$

and considered the minimization of the mean square error,

$$M = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Delta_n^2(x) \, dx$$

When this is done, one obtains the relations

$$A_k = \frac{\epsilon_k}{2\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx$$

$$B_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx \qquad (1\text{-}4)$$

The steps leading to (1-4) are left as a problem for the student. If we consider the error term, $\Delta_n(x)$, along with the relations (1-4), we find that the error term vanishes as $n \longrightarrow \infty$ and we obtain the relation

$$f(x) = \sum_{k=0}^{\infty} (A_k \cos kx + B_k \sin kx) \qquad (1\text{-}5)$$

where the coefficients A_k, B_k are given by (1-4). By comparing (1-5) with (1-1) and (1-4) with (1-3), we can see the similarity between the familiar Fourier series expansion of a function and the general orthogonal function expansion of a function. For this particular example, two sets of orthogonal

functions, i.e., sin kx and cos kx, were required in the Fourier series expansion (1-5). This led to the two sets of coefficients A_k and B_k given by (1-4). We see immediately that our assumption that we can expand $f(x)$ in terms of one set of orthogonal functions $u_i(x)$ as in (1-1) is not sufficiently general since the expansion required two sets of orthogonal functions in this particular case. We now wish to consider the problem of whether an arbitrary function can be expanded in a series of arbitrary orthogonal functions. However, the first question that must be answered is how does one determine whether a group of functions is orthogonal. The general theory that allows one to determine a set of orthogonal functions can be developed from a study of the Sturm-Liouville equation, which will be considered next.

1.2 Sturm-Liouville Equation

Most of the useful ordinary differential equations of physics, chemistry, and engineering are second order and can be represented by the general form

$$L(u) + \lambda w u = 0 \tag{1-6}$$

where the differential operator L is defined as

$$L(u) = (pu')' - qu \tag{1-7}$$

The quantities p, q,[2] and w ($w \geq 0$) are functions of x in general but λ is a constant. We shall see later that the function $u(x)$ is an eigenfunction and that the constant λ is its corresponding eigenvalue. The prime in (1-7) means the derivative with respect to x which is not necessarily the argument of the $u(x)$. Let us digress for a moment and discuss the concept of an eigenfunction. The usual linear ordinary differential equation that is considered in a lower division mathematics course has constant coefficients so that the solution consists of several terms with a number of arbitrary constants. These arbitrary constants can be determined for a specific problem by giving specific boundary conditions. However, as one considers the common differential equations encountered in the applied mathematics area, one finds that the coefficients are no longer constants and that the general solutions are often found in the form of an infinite series. Moreover, one finds that the application of specific boundary conditions for a specific problem does not result in a specific (nonzero) solution unless some of the constants in the solution take on specific values. These specific, or characteristic, values are called eigenvalues and the specific solutions are called eigenfunctions. In order to illustrate these ideas, let us consider as a simple example a vibrating string

[2]The sign of q is sometimes taken as positive rather than negative. See G. Arfken, *Mathematical Methods for Physicists*, Academic Press, Inc., New York, 1966, p. 331.

of length L, linear mass density m, and tension T. If we let x be a variable along the length of the undisturbed string and $u(x, t)$ be the disturbed displacement at the point x and at the time t, the equation of motion for u is

$$m\frac{\partial^2 u}{\partial t^2} = T\frac{\partial^2 u}{\partial x^2} \tag{1-8}$$

This is a linear second order partial differential equation and the solution may be found by the method of separation of variables. However, the equation was originally solved by d'Alembert (1747) by introducing the change of variables

$$\xi = x + \alpha_1 t, \qquad \eta = x + \alpha_2 t$$

whence (1-8) becomes

$$\frac{\partial^2 u}{\partial \xi\, \partial \eta} = 0 \tag{1-9}$$

provided $\alpha_1 + \alpha_2 = 0$ and $\alpha_1\alpha_2 + (T/m) = 0$. The solution to (1-9) is

$$\begin{aligned} u &= f(\xi) + g(\eta) \\ &= f\left(x \pm \sqrt{\frac{T}{m}}\, t\right) + g\left(x \mp \sqrt{\frac{T}{m}}\, t\right) \end{aligned} \tag{1-10}$$

The solution (1-10) represents two traveling waves since we have not yet specified any conditions concerning the ends of the string. If we impose the boundary condition that the string is fixed at its ends $x = 0, L$, and use the method of separation of variables, we obtain as the general solution to (1-8)

$$u(x, t) = [A \cos \beta t + B \sin \beta t]\left[C \cos\frac{\beta x}{\alpha} + D \sin\frac{\beta x}{\alpha}\right]$$

where $\alpha = \sqrt{T/m}$. The boundary condition that $u(0, t) = 0$ causes C to vanish. However, the boundary condition that $u(L, t) = 0$ cannot be satisfied by allowing D to vanish since then our solution would be trivial. The only remaining choice is to let $\sin(\beta L/\alpha) = 0$ which means then that

$$\frac{\beta L}{\alpha} = m\pi, \qquad m = 1, 2, \ldots$$

Thus, we find that in contrast to familiar ordinary differential equations where the boundary conditions are satisfied by choosing specific values for the constants A, B, C, D, the solution $u(x, t)$ cannot be specified unless the parameters α and β take on specific values. For each specific value of α and β, we obtain one specific solution to (1-8). The most general solution now is

an infinite series of terms as follows:

$$u(x,t) = \sum_{m=1}^{\infty} \left[A_m \cos \frac{m\pi\alpha}{L} t + B_m \sin \frac{m\pi\alpha}{L} t \right] \sin \frac{m\pi x}{L}$$

The general solution to (1-8) is the above series of eigenfunctions with an accompanying group of eigenvalues, $m\pi/L$. It also follows that solutions to (1-8) for values of α and β that do not correspond to the eigenvalues, $m\pi/L$, must be trivial (zero) solutions.[3]

Let us now return to the general equation (1-6) and show that the functions $u(x)$ are orthogonal, i.e., possess that property indicated in (1-2). We will also find that our definition of orthogonality in (1-2) is not sufficiently general to encompass all functions that satisty the Sturm-Liouville equation. Since the functions $u(x)$ satisfy the Sturm-Liouville equation, (1-6), then two specific eigenfunctions $u_i(x)$ and $u_j(x)$ with their corresponding eigenvalues λ_i and λ_j, respectively, satisfy (1-6), thus

$$\begin{aligned} L(u_i) + \lambda_i w u_i &= 0 \\ L(u_j) + \lambda_j w u_j &= 0 \end{aligned} \tag{1-11}$$

We now apply a standard procedure, which is also used to derive Green's two identities in scalar or vector form, by multiplying the first equation by u_j, the second by $-u_i$, adding the two, and then integrating over the interval (a, b) to get

$$\int_a^b [u_j L(u_i) - u_i L(u_j)]\, dx = (\lambda_j - \lambda_i) \int_a^b w u_i u_j\, dx = [p(u_j u_i' - u_i u_j')]_a^b \tag{1-12}$$

The third expression above results when one substitutes $L(u_i)$ and $L(u_j)$ from (1-7) into the first expression. The third expression vanishes because of the general boundary conditions

$$[p(u_j u_i' - u_i u_j')]_a^b = 0 \tag{1-13}$$

When written in this form, one notes that the boundary conditions necessary for orthogonality can be satisfied by a variety of conditions, e.g.,

$$\begin{aligned} p u_j u_i']_a &= p u_i u_j']_b \\ p(a) &= p(b) = 0 \\ p(a) &= 0 = u_i(b) \\ u_j(a) &= u_i(b) = 0 \\ u_i'(a) &= u_j'(b) = 0 \end{aligned}$$

[3] H. Sagan, *Boundary and Eigenvalue Problems in Mathematical Physics*, John Wiley & Sons, Inc., New York, 1961, Chap. 5.

When these boundary conditions are met, we obtain the expression of orthogonality

$$\int_a^b w u_i u_j \, dx = 0(\lambda_j - \lambda_i)^{-1}$$

As long as $\lambda_i \neq \lambda_j$, the right-hand side of the above expression vanishes. In most cases of interest, when $u_i \neq u_j$ then $\lambda_i \neq \lambda_j$ and the condition of orthogonality is valid. However, two exceptional cases of interest often occur where $u_i \neq u_j$ but $\lambda_i = \lambda_j$. In the first case, u_i and u_j are degenerate solutions of (1-6) because their eigenvalues are identical. In the second case, u_i and u_j are two linearly independent solutions of (1-6) for the same eigenvalue λ_i. This latter case leads to an expression involving the Wronskian.

If we consider the first case, where the eigenfunctions are degenerate, we find that the situation can be encountered in electromagnetic problems involving more than one dimension. For instance, the eigenfunctions, $u_{12} = \sin(\pi x/a)\sin(2\pi y/b)$ and $u_{21} = \sin(2\pi x/a)\sin(\pi y/b)$ have corresponding eigenvalues

$$\lambda_{12} = \left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{2\pi}{b}\right)^2\right]^{1/2} \quad \text{and} \quad \lambda_{21} = \left[\left(\frac{2\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right]^{1/2}$$

respectively. In general, the eigenvalues λ_{12} and λ_{21} are different unless $a = b$. When $a = b$, the eigenvalues are identical, yet $u_{12} \neq u_{21}$. Thus, the eigenfunctions are degenerate. In this case, we see that the eigenfunctions u_{12} and u_{21} are orthogonal whether λ_{12} is different from λ_{21} or not. Since the physical nature of the problem does not change wildly when the boundaries or the boundary conditions are changed slightly, we see that the difficulties introduced by degeneracy above are more mathematical deficiencies rather than some extraordinary physical phenomenon. However, the physical problem of degeneracy is quite common in the real world and often occurs in classical[4] and quantum mechanical problems[5] as well as in electromagnetic problems.[6] The situation then is that several specific states or modes may possess the same energy or occupy the same energy level.

The other exceptional case of interest occurs when $u_i \neq u_j$ because they are two linearly independent solutions of (1-6) for the same eigenvalue λ_i. In other words, u_i is not a general solution of (1-6) but one of the two linearly independent solutions U. If we let the other linearly independent solution

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[4]R. Wangsness, *Introduction to Theoretical Physics*, John Wiley & Sons, Inc., New York, 1963, p. 193.

[5]R. Wangsness, *Introductory Topics in Theoretical Physics*, John Wiley & Sons, Inc., New York, 1963, p. 232.

[6]Wangsness, *Introduction to Theoretical Physics*, p. 314.

be V, we obtain

$$\int_a^b wUV\,dx = \frac{0}{0}$$

Although, this integral is indeterminate, and of not enough interest to motivate its evaluation, let us consider the first and third expressions in (1-12) when $u_i = U$ and $u_j = V$ and the limits are ignored. Thus,

$$\int [VL(U) - UL(V)]\,dx = p(VU' - UV') = pW(V, U)$$

where $W(V, U)$ is the Wronskian[7] of V and U. If the Wronskian vanishes, U and V are not linearly independent solutions of (1-6). If we substitute the expressions for $L(V)$ and $L(U)$ from (1-7) into the left-hand side of the above integral, we obtain

$$\int [p(VU' - UV')]'\,dx = \int [pW(V, U)]'\,dx = pW(V, U) \neq 0$$

Unfortunately, this expression involving the Wronskian does not help us since we do not know the right-hand side. In fact, one might have expected the right-hand side to vanish since $\lambda_i = \lambda_j$ in the second expression in (1-11). However, this difficulty is caused by our implicit assumption that $\lambda_i \neq \lambda_j$ in (1-11). When $\lambda_i = \lambda_j$ in (1-11) the second expression in (1-12) does not appear since we obtain, for $u_i = U$ and $u_j = V$,

$$VL(U) - UL(V) = 0 = [pW(V, U)]'$$

and thus,

$$pW(V, U) = c \tag{1-14}$$

where c is a constant with respect to x. The reason we mention the difficulty encountered above is to emphasize the fact that the Sturm-Liouville equation is not particularly useful when evaluating the Wronskian or when evaluating the norm. The evaluation of the norm is very similar to the case we just discussed except that now either $u_i = u_j = U = V$ or $u_i = u_j = AU + BV$. In this case we find that the expression for the norm, (1-2), is not sufficiently general since the second expression in (1-12) gives us the correct general expression for the norm

$$\int_a^b wu_i^2\,dx = N$$

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[7] H. Margenau and G. Murphy, *The Mathematics of Physics and Chemistry*, D. Van Nostrand Company, Inc., Princeton, N.J., 1943, p. 130.

and the general orthogonality expression

$$\int_a^b w u_i u_j \, dx = N\delta_{ij} \tag{1-15}$$

If we consider the first expression in (1-12) again, we find that

$$\int_a^b u_j L(u_i) \, dx = \int_a^b u_i L(u_j) \, dx, \qquad i \neq j$$

and the operator L is called a Hermitian operator. It is also possible for the eigenfunctions u_i and u_j to be complex, in which case the expression above and (1-11) must be modified. However, the eigenvalues λ_i and λ_j remain real. Further details concerning self-adjoint operators and Hermitian operators may be found elsewhere.[8]

1.3 Orthogonal Functions

In this section we consider some of the orthogonal functions that appear frequently in electromagnetic theory. It is assumed that the student is familiar with these functions, the various differential equations that they satisfy, and the general procedure for solving the equations. Thus, the functions can be introduced in terms of their generating functions. This alternate representation for the functions is usually very advantageous when deriving recurrence relations and other useful identities.

The first functions that we will discuss are the Bessel functions. These functions occur in problems involving heat flow, fluid flow, and wave motion provided solutions are being sought in the cylindrical coordinate system. The thing that is common to all of the above problems is that the spatial dependence of the function can be expressed in terms of the Laplacian. As a result, one can state that the Laplacian in cylindrical coordinates can be separated into ordinary differential equations, one of which is generally Bessel's equation. However, Bessel's equation also occurs in connection with other problems,[9] many of which still involve the Laplacian in different coordinate systems.[10] On the other hand, there are a great many ordinary differential equations that can be transformed by a change of variables into Bessel's equation so that their solutions can be expressed in terms of Bessel func-

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[8] *Ibid.*, pp. 255, 328. Also, see Arfken, *Mathematical Methods for Physicists*, Chap. 9.

[9] H. Reddick and F. Miller, *Advanced Mathematics for Engineers*, 2nd ed., John Wiley & Sons, Inc., New York, 1947.

[10] Arfken, *Mathematical Methods for Physicists*, Chap. 8.