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Intermediate Real Analysis

With 100 Illustrations

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Preface

There are a great deal of books on introductory analysis in print today, many written by mathematicians of the first rank. The publication of another such book therefore warrants a defense. I have taught analysis for many years and have used a variety of texts during this time. These books were of excellent quality mathematically but did not satisfy the needs of the students I was teaching. They were written for mathematicians but not for those who were first aspiring to attain that status. The desire to fill this gap gave rise to the writing of this book.

This book is intended to serve as a text for an introductory course in analysis. Its readers will most likely be mathematics, science, or engineering majors undertaking the last quarter of their undergraduate education. The aim of a first course in analysis is to provide the student with a sound foundation for analysis, to familiarize him with the kind of careful thinking used in advanced mathematics, and to provide him with tools for further work in it. The typical student we are dealing with has completed a three-semester calculus course and possibly an introductory course in differential equations. He may even have been exposed to a semester or two of modern algebra. All this time his training has most likely been intuitive with heuristics taking the place of proof. This may have been appropriate for that stage of his development. However, once he enters the analysis course he is subject to an abrupt change in the point of view and finds that much more is demanded of him in the way of rigorous and sound deductive thinking. In writing the book we have this student in mind. It is intended to ease him into his next, more mature stage of mathematical development.

Throughout the text we adhere to the spirit of careful reasoning and rigor that the course demands. We deal with the problem of student adjustment to the stricter standards of rigor demanded by slowing down the pace at which topics are covered and by providing much more detail in the proofs than is customary in most texts. Secondly, although the book contains its share of abstract and general results, it concentrates on the specific and concrete by applying these theorems to gain information about some of the important functions of analysis. Students are often presented and even have proved for them theorems of great theoretical significance without being given the opportunity of seeing them "in action" and applied in a non-trivial way. In our opinion, good pedagogy in mathematics should give substance to abstract and general results by demonstrating their power.

This book is concerned with real-valued functions of one real variable. There is a chapter on complex numbers, but these play a secondary role in the development of the material, since they are used mainly as computational aids to obtain results about trigonometric sums.

For pedagogical reasons we avoid "slick" proofs and sacrifice brevity for straightforwardness.

The material is developed deductively from axioms for the real numbers. The book is self-contained except for some theorems in finite sets (stated without proof in Chapter II) and the last theorem in Chapter XIV. In the main, any geometry that is included is there for purposes of visualization and illustration and is not part of the development. Very little is required from the reader in the way of background. However, we hope that he has the desire and ability to follow a deductive argument and is not afraid of elementary algebraic manipulation. In short, we would like the reader to possess some "mathematical maturity." The book's aim is to obtain all its results as logical consequences of the fifteen axioms for the real numbers listed in Chapter I.

The material is presented sequentially in "theorem–proof–theorem" fashion and is interspersed with definitions, examples, remarks, and problems. Even if the reader does not solve all the problems, we expect him to read each one and to understand the result contained in it. In many cases the results cited in the problems are used as proofs of later theorems and constitute part of the development. When the reader is asked, in a problem, to prove a result which is used later, this usually involves paralleling work already done in the text.

Chapters are denoted by Roman numerals and are separated into sections. Results are referred to by labeling them with the chapter, section, and the order in which they appear in the section. For example, Theorem X.6.2 refers to the second theorem of section 2 in Chapter X. When referring to a result in the same chapter, the Roman numeral indicating the chapter is omitted. Thus, in Chapter X, Theorem X.6.2 is referred to as Theorem 6.2.

We also mention a notational matter. The open interval with left endpoint a and right endpoint b is written in the book as $(a;b)$ using a semi-colon between a and b, rather than as (a, b). The latter symbol is reserved for the ordered pair consisting of a and b and we wish to avoid confusion.

I owe a special debt of gratitude to my friend and former colleague Professor Abe Shenitzer of York University in Ontario, Canada, for patiently reading through the manuscript and editing it for readability.

My son Joseph also deserves special thanks for reading most of the material, pointing out errors where he saw them, and making some valuable suggestions.

Thanks are due to Professors Eugene Levine and Ida Sussman, colleagues of mine at Adelphi University, and Professor Gerson Sparer of Pratt Institute, for reading different versions of the manuscript.

Ms. Maie Croner typed almost all of the manuscript. Her skill and accuracy made the task of readying it for publication almost easy.

I am grateful to the staff at Springer-Verlag for their conscientious and careful production of the book.

To my wife Sylvia I give thanks for her patience through all the years the book was in preparation. תושלב״ע

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Garden City, L. I., N. Y.
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E. F.

Contents

CHAPTER I
Preliminaries

1. Sets

We think of a *set* as a collection of objects viewed as a single entity. This description should not be regarded as a definition of a set since in it "set" is given in terms of "collection" and the latter is, in turn, in need of definition. Let us rather consider the opening sentence merely as a guide for our intuition about sets.

The objects a set consists of are called its *members* or its *elements*. When S is a set and x is one of its members we write $x \in S$, and read this as "x belongs to S" or as "x is a member of S" or as "x is an element of S." When $x \in S$ is false, we write $x \notin S$.

To define a set whose members can all be exhibited we list the members and then put braces around the list. For example,

$$M = \{\text{Peano, Dedekind, Cantor, Weierstrass}\}$$

is a set of mathematicians. We have Cantor $\in M$, but Dickens $\notin M$.

When a set theory is applied to a particular discipline in mathematics, the elements of a set come from some fixed set called the *domain of discourse*, say U. In plane geometry, for example, the domain of discourse is the set of points in some plane. In analysis, the domain of discourse may be $\mathbf{R}$, the set of real numbers, or $\mathbf{C}$, the set of complex numbers.

As an intuitive crutch, it may help to picture the domain of discourse U as a rectangle in the plane of the page and a set S in this domain as a set of points bounded by some simple closed curve in this rectangle (Fig. 1.1). The figure suggests that $x \in S$, but $y \notin S$.

A *singleton* is a set consisting of exactly one member as in $A = \{b\}$. We have

$$x \in \{b\} \quad \text{if and only if} \quad x = b. \tag{1.1}$$

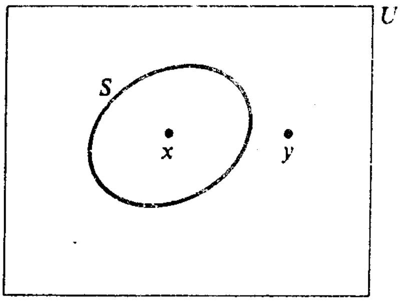


Figure 1.1

We distinguish between the set $\{b\}$ and its member b. Thus, we write

$$b \neq \{b\} \qquad \text{for each } b. \tag{1.2}$$

For example, 2 is a number, but $\{2\}$ is a certain set of numbers.

$S = \{a, b\}$ is a set whose members are a and b. We refer to it as the *unordered pair* consisting of a and b.

Unfortunately, the sets usually dealt with in mathematics are such that their members cannot all be exhibited. Therefore, we describe sets by means of a property common to all their members. Let $P(x)$ read "x has property P." The set B of elements having property P is written

$$B = \{x \mid P(x)\}. \tag{1.3}$$

This is read as "B is the set of all x such that x has property P." For example, the set $\mathbf{R}$ of real numbers will be written

$$\mathbf{R} = \{x \mid x \text{ is a real number}\}. \tag{1.4}$$

Here, $P(x)$ is the sentence "x is a real number." If U is the domain of discourse, the set of members of U having property P is often written

$$B = \{x \in U \mid P(x)\}. \tag{1.5}$$

This is read as "the set of all x belonging to U such that x has property P."

A set A is called a *subset* of a set B and we write $A \subseteq B$, if and only if each element of A is an element of B, i.e., if and only if $x \in A$ implies $x \in B$. We visualize this in Fig. 1.2. Each part of this figure suggests that

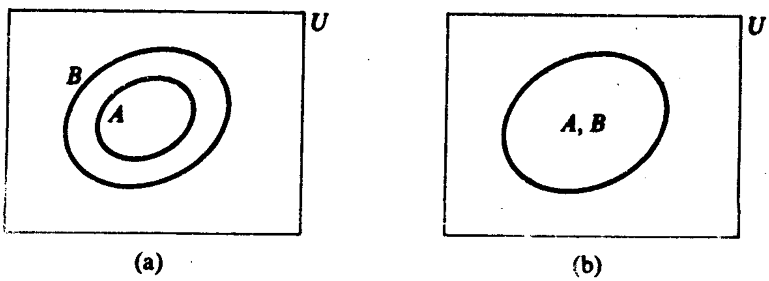
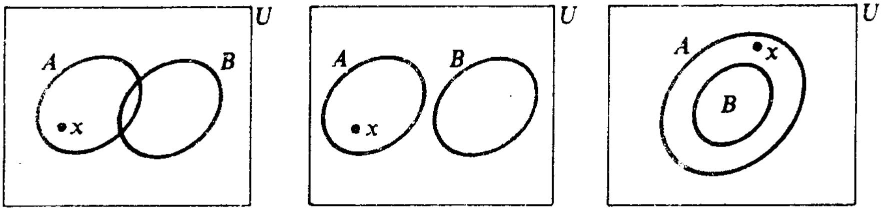


Figure 1.2



(a) (b) (c)

Figure 1.3

each element of A is an element of B; in (a) there are supposed to be elements of B not in A, where as in (b) every element of B is also an element of A. Thus, $A \subseteq B$ holds also when A and B have the same members. If $A \subseteq B$ is false, we write $A \not\subseteq B$.

$$A \not\subseteq B \text{ is equivalent to "there exists } x \in A \text{ such that } x \notin B." \qquad (1.6)$$

Each of the diagrams in Fig. 1.3 portrays the situation $A \not\subseteq B$.

Sets A and B are called *equal* and we write $A = B$, if and only if both

$$A \subseteq B \quad \text{and} \quad B \subseteq A$$

hold. Thus, $A = B$, if and only if A and B have the same members.

When A and B are sets such that $A \subseteq B$ but $A \neq B$, we call A a *proper* subset of B and write

$$A \subset B.$$

This means that every element of A is an element of B, but there exists an $x \in B$ such that $x \notin A$.

One should distinguish carefully between the notions "$\in$" and "$\subseteq$." Thus, $x \in A$ means that x is an element of A, while $A \subseteq B$ means that $x \in A$ implies $x \in B$. The distinction is perhaps more noticeable when we deny these relations. For example, $x \notin A$ means x is not a member of A, whereas $A \not\subseteq B$ means that there exists $x \in A$ such that $x \notin B$. The distinction is important. The two relations have different properties. Thus, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$ (cf. Prob. 1.3). Because of this $\subseteq$ is called a *transitive relation*. On the other hand, the relation "$\in$" is not transitive. For consider

$$X = 1, \quad A = \{1\}, \quad \text{and} \quad B = \{\{1\}\}.$$

Here B is a singleton set whose member is $A = \{1\}$ (nothing prevents us from having sets whose members are sets). We have $X \in A$ and $A \in B$, but $X \in B$ is false since this would imply $X = \{1\}$ or $1 = \{1\}$ and this is false (cf. (1.2)).

When $x \in A$ and $A \subseteq B$, we write $x \in A \subseteq B$. This clearly implies $x \in B$. Similarly, when $A \subseteq B$ and $B \subseteq C$, we write $A \subseteq B \subseteq C$.

A set having no members is said to be *empty*. Such a set is also called a *null* set. Sometimes, in the course of a mathematical discussion, a set is defined by some property. When no elements exist which have this property we call the set *empty*. An empty set is written $\emptyset$. We prove that for any set A we have

$$\emptyset \subseteq A. \tag{1.7}$$

Were this false, i.e., $\emptyset \not\subseteq A$, there would exist $x \in \emptyset$ such that $x \notin A$. This is impossible since no $x \in \emptyset$ exists.

PROB. 1.1. Prove: $\{a,b\} = \{a,b,a\}$.

PROB. 1.2. Prove: If A is a set, then $A \subseteq A$ and $A = A$.

PROB. 1.3. Prove: If $A \subseteq B \subseteq C$, then $A \subseteq C$.

PROB. 1.4. Prove: If A and B are sets, then $A = B$ implies $B = A$.

PROB. 1.5. When A, B, and C are sets such that $A = B$ and $B = C$, we write $A = B = C$. Prove: $A = B = C$ implies $A = C$.

PROB. 1.6. Prove: If $A \subset B \subseteq C$ or $A \subseteq B \subset C$, then $A \subset C$.

PROB. 1.7. Prove: If $A \subseteq \emptyset$ for some set A, then $A = \emptyset$ (cf. (1.7)).

Remark 1.1. Examine the sets $A = \{a,b\}$ and $B = \{b,c\}$, where a, b, and c are distinct. Clearly $A \not\subseteq B$ and $B \not\subseteq A$ hold. Thus, not all sets are related by the subset relation.

2. The Set $\mathbb{R}$ of Real Numbers

We shall treat the real numbers axiomatically and list 15 axioms for them. In this section we cite only 14 of the 15 axioms. The fifteenth axiom will be called the *completeness* axiom and will be stated in Section 8.

The set $\mathbb{R}$ of real numbers is postulated to have the properties:

(I) **(Axiom 0_1).** There are at least two real numbers.

(II) **(Axiom 0_2).** There is a relation called *less than*, written as $<$, between real numbers such that if x and y are real numbers, then exactly one of the following alternatives holds: Either (1) $x = y$ or (2) $x < y$ or (3) $y < x$;

PROB. 2.1. Prove: If x is a real number, then $x < x$ is false.

[We need not postulate the existence of a *greater than* relation between real numbers since this relation can be defined in terms of the "less than" relation.]

Def. 2.1. We define $x > y$ to mean $y < x$, reading this as "x is greater than y." We can now reformulate Axiom 0_2 as

(II′) **(Axiom $0_2'$).** If x and y are real numbers, then exactly one of the following alternatives holds: Either (1) $x = y$ or (2) $x < y$ or (3) $x > y$.

Def. 2.2. When $x < y$ or $x = y$, we write $x \leqslant y$.

PROB. 2.2. Prove: If x and y are real numbers such that $x \leqslant y$ and $x \geqslant y$, then $x = y$.

(III) **(Axiom 0_3).** If x, y and z are real numbers such that $x < y$ and $y < z$, then $x < z$.

Def. 2.3. When $x < y$ and $y < z$ both hold, we write $x < y < z$. Thus, by Axiom 0_3, $x < y < z$ implies $x < z$.

PROB. 2.3. Prove: (a) Either of $x < y \leqslant z$ or $x \leqslant y < z$ imply $x < z$; (b) $x \leqslant y \leqslant z$ implies $x \leqslant z$.

[We now introduce postulates for addition and multiplication. The lowercase Latin letters x, y, z appearing in the axioms below will represent real numbers.]

(IV) **(Axiom A_1)** (Closure for Addition). If x and y are real numbers, there is a unique real number $x + y$ called the *sum* of x and y.

(V) **(Axiom A_2)** (Associativity for Addition)

$$(x + y) + z = x + (y + z); \tag{2.1}$$

(VI) **(Axiom A_3)** (Commutativity for Addition)

$$x + y = y + x; \tag{2.2}$$

[The next axiom relates addition to the "less than" relation in R.]

(VII) **(Axiom 0_4).** $x < y$ implies $x + z < y + z$.

PROB. 2.4. Prove: $x < y$ and $u < v$ imply $x + u < y + v$.

(VIII) **(Axiom S).** If x and y are real numbers, there is a real c such that $y + c = x$;

(IX) **(Axiom M_1)** (Closure for Multiplication). If x and y are real numbers, there is a real number xy (also written as $x \cdot y$) called the *product* of x and y;

(X) **(Axiom M_2)** (Associativity for Multiplication)

$$(xy)z = x(yz). \tag{2.3}$$

(XI) **(Axiom M_3)** (Commutativity for Multiplication)

$$xy = yx. \tag{2.4}$$

(XII) **(Axiom D)** (Distributive Law)

$$x(y + z) = xy + xz. \tag{2.5}$$

[The next axiom relates multiplication to the "less than" relation in $\mathbb{R}$.]

(XIII) **(Axiom 0_5).** $x < y$ and $u < v$ imply $xu + yv > xv + yu$.

(XIV) **(Axiom Q).** If x and y are real numbers, where $z + y \neq z$ holds for some real z, then there exists a real number q such that $yq = x$.

Thus far, 14 axioms were cited. As mentioned earlier, the fifteenth and last one will be stated in Section 8.

The axioms indicate that addition and multiplication are *binary operations*, that is, we add and multiply two numbers at a time. We define $x + y + z$ and xyz by means of

$$x + y + z = (x + y) + z \quad \text{and} \quad xyz = (xy)z. \tag{2.6}$$

Axioms A_2 and M_2 respectively imply that

$$x + y + z = x + (y + z) \quad \text{and} \quad xyz = x(yz). \tag{2.7}$$

Having defined $x + y + z$ and xyz, we define $x + y + z + u$ and $xyzu$ as

$$\begin{aligned} x + y + z + u &= (x + y + z) + u, \\ xyzu &= (xyz)u. \end{aligned} \tag{2.8}$$

Prob. 2.5. Prove: If x, y, z and u are real numbers, then
(a) $x + y + z + u = (x + y) + (z + u) = x + (y + z + u)$ and
(b) $xyzu = (xy)(zu) = x(yzu)$.

Prob. 2.6. Prove: (a) $x + z < y + z$ implies $x < y$
(b) $x + z = y + z$ implies $x = y$. The result in part (b) is called the *cancellation law* for addition.

Prob. 2.7. Prove: The c such that $y + c = x$, of Axiom S, is unique.

Theorem 2.1. *There exists a real number z such that $x + z = x$ holds for all $x \in \mathbb{R}$. This z is the only real number with this property.*

Proof. Let b be some real number. By Axiom S, there exists a real z such that $b + z = b$. We prove that $x + z = x$ for *all* $x \in \mathbb{R}$. From $b + z = b$, we obtain for $x \in \mathbb{R}$,

$$(b + z) + x = b + x \quad \text{and hence,} \quad b + (z + x) = b + x. \tag{2.9}$$

In the second equality, we "cancel" the b on both sides to obtain $z + x = x$. This proves the existence of z. Next we prove its uniqueness.

Assume that there also exists a $z' \in \mathbb{R}$ such that $x + z' = x$ for all $x \in \mathbb{R}$. It follows that $z + z' = z$. Similarly, in view of the property of z, $z' + z = z'$. By Axiom A_3 we have $z + z' = z' + z$ and we conclude that $z' = z$. This completes the proof.

Def. 2.4. The z in $\mathbb{R}$ such that $x + z = x$ for all $x \in \mathbb{R}$ is called *zero* and is written as 0. Thus

$$x + 0 = x = 0 + x \qquad \text{for all} \quad x \in R. \tag{2.10}$$

Theorem 2.2. *If* $x \in \mathbb{R}$, *then* $x0 = 0$.

PROOF. For any y in $\mathbb{R}$

$$xy + x0 = x(y + 0) = xy = xy + 0,$$

so that $xy + x0 = xy + 0$. "Cancelling" xy we obtain $x0 = 0$ as claimed.

Given $x \in \mathbb{R}$, there exists (Axiom S) a real y such that $x + y = 0$. y is the only real number with this property (why?).

Def. 2.5. For each $x \in \mathbb{R}$, the unique y such that $x + y = 0$ is called the *negative* (or *additive inverse*) of x and is written as $-x$. Accordingly, we have

$$x + (-x) = 0 \qquad \text{for each} \quad x \in R. \tag{2.11}$$

PROB. 2.8. Prove: $-0 = 0$.

PROB. 2.9. Prove: $-(-x) = x$ for each $x \in R$.

Def. 2.6. Define $x - y$ as the c such that $y + c = x$ and call it x *minus* y.

PROB. 2.10. Prove: (a) $y + (x - y) = x$ and (b) $x - y = x + (-y)$.

PROB. 2.11. Prove: $-(x - y) = y - x$.

PROB. 2.12. Prove: $z + y \neq z$ if and only if $y \neq 0$.

Def. 2.7. A real p such that $p > 0$ is called *positive*. A real n such that $n < 0$ is called *negative*.

PROB. 2.13. Prove: If $x > 0$ and $y \geqslant 0$, then $x + y > 0$.

PROB. 2.14. Prove: If $x > y$, then (a) $z > 0$ implies $xz > yz$ and (b) $z < 0$ implies $xz < yz$. (Hint: use Axiom 0_5)