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# STUDIES IN OPTIMIZATION

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Professor Tucker's contribution, "Combinatorial Algebra of Linear Programs," is a slight revision of a paper, prepared with the assistance of M. Balinski, H. D. Mills, and R. R. Singleton, and published in NEW DIRECTIONS IN MATHEMATICS (Dartmouth College Mathematics Conference, 1961), edited by J. G. Kemeny, R. Robinson, and R. W. Ritchie, Prentice-Hall 1963, pp. 77–91; Professor Cottle's and Professor Dantzig's contribution, "Complementary Pivot Theory of Mathematical Programming," previously appeared in MATHEMATICS OF THE DECISION SCIENCES, Part 1, Lectures in Applied Mathematics, Volume 11, published by AMS, 1968, and appeared also in LINEAR ALGEBRA AND ITS APPLICATIONS, Volume 1, 1968; Professor Edmonds' and Professor Fulkerson's contribution, "Bottleneck Extrema," previously appeared in the JOURNAL OF COMBINATORIAL THEORY, Volume 8, Number 3, 1970; Professor Scarf's and Dr. Shapley's contribution, "On Cores and Indivisibility," previously appeared in the JOURNAL OF MATHEMATICAL ECONOMICS, Volume 1, 1974; Professor Dantzig's and Dr. Wolfe's contribution, "The Decomposition Algorithm for Linear Programs," previously appeared in ECONOMETRICA, Volume 29, Number 4, 1961.

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# INTRODUCTION

This volume of the Mathematical Association of America series Studies in Mathematics is a collection of eight papers from the field of mathematical optimization. This collection constitutes an interesting sample of topics and techniques of current interest in optimization, however, we caution, it is too small to be considered representative of this important and developing subject.

As practiced, mathematical optimization can be approximately described as the tasks of

(1) Developing a mathematical structure, called a program, which models some "real world situation"; in general, relations of the structure represent restrictions on the value of the variables and the objective function(s) provides a measure(s) of performance.
(2) Investigating existence and attributes of optimal (or near optimal) solutions; finding ways to characterize optimal policies.
(3) Designing and utilizing algorithms for computation of optimal (or near optimal) solutions.
(4) Implementing the mathematical solution in a particular application, evaluating the results, and making modifications.

Each paper of this volume treats some aspects of tasks (1), (2) and (3). After making brief comments about each paper, we will say a few words about the relationship of optimization theory to an education in mathematics.

The first paper by A. W. Tucker, "Combinatorial Algebra of Linear Programs" opens and closes with discussions regarding the growing impact of combinatorics in applied areas of mathematics. As linear programming is perhaps the main prerequisite for appreciating the other articles in this volume, a reader might well begin his perusing with this article.

One device for studying a mathematical program is through duality theory. Given an optimization problem, called the *primal* program, there is often another optimization problem, called the *dual*, which is related in a very strong and intricate way to the original problem; that is, to the primal program. For example, the optimal objective value of the primal and dual programs might be equal or a solution to the dual might yield a solution to the primal. Most of the papers in this volume, e.g., that of Tucker, make extensive use of duality concepts.

The second paper by Richard W. Cottle and George B. Dantzig, "Complementary Pivot Theory of Mathematical Programming", defines the linear complementarity problem as that of seeking $n$-vectors $w$ and $z$ satisfying:

$$w = q + Mz \qquad w \geqq 0 \qquad z \geqq 0$$
$$w \cdot z = 0$$

where $M$, a square matrix, and $q$, a vector, are given. This model is shown to include linear and quadratic programming and general sum two-person games in normal form. Lemke's finite complementary pivoting algorithm for finding a complementary solution for certain $M$ and $q$ is next described and established. As the simplex method for linear programming the scheme iterates by moving from one extreme point to the next along edges of a polyhedral set. However, Lemke's algorithm is most unusual in that convergence is not based on monotone improvement of some function. It is this feature that permits one to find globally optimal solutions to certain classes of non-linear non-convex extremum problems.

Harold W. Kuhn, in his article, " 'Steiner's' Problem Revisited", treats the problem of finding a point, popularly called a Steiner Point, in Euclidean space such that the weighted sum of distances from this point to $n$ given points is minimum. The Steiner Point is shown to have certain attributes. The non-linear dual problem is developed and shown to have an interesting form. An algorithm for solving the problem is described and its convergence to the unique solution is proved.

The article by B. Curtis Eaves, "Properly Labeled Simplexes", treats a generalization of Sperner's Lemma; triangulations are needed in the proof but not in the statement of this result. The unique convergence proof of Lemke for complementary pivoting, as discussed in the article of this volume by Cottle and Dantzig, is employed on an infinite structure. An algorithm for computing fixed points is a natural by-product of the method of proof.

Jack Edmonds and D. R. Fulkerson, in their article, "Bottleneck Extrema", develop a "min max" problem

$$\min_{R \in \mathcal{R}} \max_{x \in R} f(x)$$

wherein they minimize over certain subsets $R$ of a finite set $E$ and maximize over elements $x$ of the subsets $R$. For example, one might consider the task of finding a route through a network between two given points such that the bottleneck (that is, the least arc number encountered on the route) is least restrictive. A dual "max min" problem

$$\max_{S \in \mathcal{S}} \min_{x \in S} f(x)$$

is treated wherein they maximize over certain subsets of the same finite set $E$ and minimize over elements of the subset. The main results are a theorem that the primal and dual problems have equal objective value and an algorithm for computing solutions to these two problems. Note that the duality theorem here, for example, is based on a discrete or combinatorial structure whereas others, as in Kuhn's paper, are based on continuous or nondiscrete structures.

Herbert Scarf and Lloyd S. Shapley, in their article, "On Cores and Indivisibility", develop a model for a market in which $n$ participants trade indivisible goods; each participant (a trader) enters the market with one unit of a good, say a house, and his preference ordering of all goods of the market. It is proved that the "core" is nonempty, that is, it is proved that there is a scheme of trading among the $n$ participants such that no subcollection of the participants, in view of their preferences, might wish to withdraw to trade only among themselves. The development is based upon a theorem of Scarf; a proof of this theorem is given in Example 4 of "Properly Labeled Simplexes" in this volume.

The article by Arthur F. Veinott, Jr., "Markov Decision Chains", treats the problem of optimally controlling a finite state Markov chain process. If the process is initiated in a given state, then one of a finite number of alternatives must be selected which results in an "immediate income" (or expense) but also determines the probability distribution of the transition to the next state. An infinite sequence of immediate incomes is thus generated. The paper concerns itself with evaluating these income sequences, under various criteria, investigating the properties of optimal solutions, and computing optimal solutions. Applications include a gambling problem, a sequential decision problem, and an inventory problem.

The last paper, "The Decomposition Algorithm for Linear Programs", by George B. Dantzig and Philip Wolfe, provides a device whereby one large linear program is partitioned into two sets, one representing a set of "joint" constraints and the second set consisting of a collection of smaller subproblems that are independent of each other except that their variables are related through the joint constraints. Following partition one has associated with the joint constraints a "master" program which coordinates the solutions of a collection of subproblem programs. The coordination between the master and subproblems proceeds roughly as follows: the master delivers to a subproblem tentative "prices" (Lagrange Multipliers) associated with the joint constraints; assuming these prices, the subproblem optimizes and returns to the master problem certain facts about this optimal solution. Using this new information the master linear program is resolved. The process repeats and terminates with an optimal solution to the original large linear program. This decomposition principle "yields a certain rationale for the decentralized decision process in the theory of the firm."

The remainder of this introduction is devoted to a recommendation regarding the future of optimization theory in mathematics education.

Optimization as we have described it became viable with the advent of computers. The computer revolution has so broadened the base of quantitative analysis that almost all areas of human endeavor are now being modelled in mathematical terms. It is said that approximately one-fourth of all current scientific computation involves optimization. It is this force that has spurred the rapid progress in this field.

Optimization theory is now a fertile ground for new and pressing problems, for classes of problems upon which to build new mathematical theories; it could be a source of new, exciting, and relevant problems which would serve to stimulate and to motivate the mathematics student. The creative student could be challenged by the collection of outstanding unsolved problems. Two such problems are the traveling salesman problem and the Hirsh (or $m$-step) conjecture; both of these problems are concerned with finding a "good algorithm" in the sense of J. Edmonds: a good algorithm is one in which the time to compute a solution to a numerical problem grows algebraically (e.g., not exponentially) with the size of the problem.

The shortest route problem is related to the traveling salesman problem but is much easier: given a road map, consider the task of finding the shortest distance from San Francisco to Boston. One formulates this problem in mathematical terms by representing the cities as points in a set and roads between cities by arcs, a binary relation. A distance is assigned to the arcs and the concept of a simple path in a graph is introduced. Since the number of simple paths between any two points is obviously finite, the shortest route problem is uninteresting from an existence point of view. Namely, just pick the shortest path among the finite set of possible paths.

Unfortunately for the case of a complete graph with $n$ nodes there are many such paths. Using direct enumeration for $n = 90$, if there were $10^{30}$ computers on each of the $10^{11}$ stars of the Milky Way, if each computer analyzed $10^{30}$ paths per second, and if all these computers were running in parallel since the beginning of time $10^{10}$ years ago (the big bang), only a tiny fraction of the paths would, to date, have been examined. Given $m$ nodes and $n$ arcs, can one devise a shortest route algorithm that requires at most, say $n + m$ additions and comparisons; what is the best one can do? It is known that $n^2/2$ additions and $n^2$ comparisons can be attained for the complete graph with non-negative distances. Thus for the shortest route problem good algorithms have been devised provided the arc distances are non-negative. If the arc-distances can have negative as well as positive values does there exist a "good" shortest path algorithm? The enlarged class of problems now includes the famous traveling salesman problem: a traveling salesman has a sweetheart in the capital city of each of the 50 states. He feels it is his duty to arrange his tour so as to visit each of his sweethearts. Find the shortest tour. This problem has been lying about in mathematical circles since the 1930's and perhaps earlier. From an algorithmic point of view this is a fascinating problem.

It is now known that the simplex method of linear programming is not a "good algorithm" in the sense of Edmonds. Nevertheless, for tens of thousands of applied problems the simplex method has converged in the neighborhood of $3m$ steps where $m$ is the number of equations in the linear program. However, examples have been constructed where the number of iterations grows exponentially with $n$, the number of variables. A full explanation of the difference between practiced and contrived examples appears to be beyond the capabilities of current mathematics. An interesting problem related to this general question is the Hirsh (or $m$-step) conjecture. Namely, given two extreme points $p$ and $q$ of an $n$-dimensional polyhedral convex set $P$ with $f$ full dimensional faces; does there exist a path of extreme edges from $p$ to $q$ with no more than $m = f - n$ edges? This existence question is easy enough to understand, as is the problem of the traveling salesman, and as elusive.

It seems to us that omission of optimization in the typical mathematics curriculum deprives the student of an exciting and relevant outlet for their talents. Recently a combinatorial course at a major university was informally surveyed to find out how many had even heard of such subjects as linear programming, mathematical programming, network theory, or integer programming. Only about half the class had. Those who had not were mathematics majors.

In order to expose the student to mathematical optimization we have two suggestions. *First*, new courses could supplement (or replace) those in the traditional undergraduate curriculum in mathematics; for example, an introductory course on finite mathematics (already popular in many schools) or an advanced undergraduate course on combinatorial analysis which includes some optimization theory. Classical combinatorics is sometimes useful—it occasionally prevents people from programming an exhaustive search procedure on the computer. For example, one would probably decide not to list out all the paths in a network, if he knew in advance that there were an astronomical number of possibilities. However, the portion of combinatorial analysis which seems to have the most important applications is that concerned with selecting the best of all the combinations. This, for example, is what linear programming is all about. An economy has many alternative technologies it can draw upon; some use more labor than others and make more intense use of scarce resources, or more intensive use of limited capacity. The problem then becomes how to select, how much, and when.

*Second*, aspects of university organization might be revised so that all students of mathematics are more likely to be exposed, as a matter of curricular routine, to the "mathematical sciences". At present, there is no general descriptive term to cover the fields of operations research, management science, control theory, statistics, numerical analysis as applied to computer science, and traditional applied mathematics. What is emerging instead is one sweeping descriptive term "Mathematical Sciences" which also includes "pure" mathematics. Universities might consider developing a

School of Mathematical Sciences encompassing mathematics, operations research, statistics, computer science, and classical applied mathematics.

As an historical note, a number of fields would be broadly referred to as belonging to "Applied Mathematics" except for a semantic difficulty: the term "Applied Mathematics" has been used, traditionally, for mathematical systems drawn from the physical sciences. New terminology has emerged to circumvent this semantic impasse, for example "Mathematical Sciences" and "Operations Research."

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# COMBINATORIAL ALGEBRA OF LINEAR PROGRAMS*

*A. W. Tucker*

Many advances on the frontiers of mathematics are related increasingly to the "applied" interests of social scientists, economists, statisticians, operations researchers, industrial and design engineers, and others. Such subjects as linear programming, game theory, network theory, Boolean algebra, Markov processes, and information theory are used frequently in industrial and government applications, and appear increasingly often both as topics for research and as tools in investigations in other areas. These and other related subjects fall mainly within the rapidly developing field of Combinatorial Mathematics. The author's view of the importance of this field was given in the Foreword to the Novem-

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* This is a slight revision of a paper, prepared with the assistance of M. L. Balinski, H. D. Mills, and R. R. Singleton, and published in NEW DIRECTIONS IN MATHEMATICS (Dartmouth College Mathematics Conference, 1961), edited by J. G. Kemeny, R. Robinson and R. W. Ritchie, Prentice-Hall 1963, pp. 77–91.

ber, 1960, issue of the *IBM Journal*, which was especially devoted to the field.

"Combinatorial Mathematics, or 'Combinatorics', regarded as originating in the *Ars Combinatoria* of Leibniz, has to do with problems of arrangement, operation, and selection within a finite or discrete system—such as the aggregate of all possible states of a digital computer. Until recently, preoccupation with continuous mathematics has inhibited the growth of discrete mathematics. But now it is realized that combinatorial methods can be developed to attack profitably, in modern science and technology, a vast variety of 'problems of organized complexity'—an apt designation of Warren Weaver [1]. In 1947 Hermann Weyl [2] wrote as follows (rearranged slightly for quotation here):

" 'Perhaps the philosophically most relevant feature of modern science is the emergence of abstract symbolic structures as the hard core of objectivity behind—as Eddington puts it—the colorful tale of the subjective storyteller mind. The combinatorics of aggregates and complexes deals with some of the simplest such structures imaginable. It is gratifying that combinatorial mathematics is so closely related to the philosophically important problems of individuation and probability, and that it accounts for some of the most fundamental phenomena in inorganic and organic nature. This structural viewpoint occurs in the foundations of quantum mechanics. In a widely different field John von Neumann's and Oskar Morgenstern's attempt to found economics on a theory of games is characteristic of the same trend. The network of nerves joining the brain with the sense organs is a subject that by its very nature invites combinatorial investigation. Modern computing machines translate our insight into the combinatorial structure of mathematics into practice by mechanical and electronic devices.' "

Let us now examine one specific new direction in applied mathematics which exhibits combinatorial structure characteristic of the field outlined above. This is *linear programming,* a subject born in 1947 and now extended in various ways under the title "mathematical programming"—to avoid confusion with computer programming. This subject has an extensive literature from which four items are selected rather arbitrarily for mention:—a complete treatise [3], with introductory chapters on problems, origins, and models, by the originator of the subject, and three elementary textbooks, [4], [5], and [6], with linear-programming chapters using the same tableau-pivot format we do here.

## A MINIATURE EXAMPLE OF LINEAR PROGRAMMING

A *linear program* is a problem of "optimizing" (i.e., maximizing or minimizing) a linear "objective" function of many (real) variables subject to a system of linear "constraints," each of which is a linear inequality or linear equation. The following is an example, in miniature,

Minimize the objective function

$$(0) \quad -\lambda + 3\mu$$

subject to the constraints

$$(1) \quad -\lambda + 2\mu \geqq 2$$

$$(2) \quad \lambda - \mu \geqq -3$$

$$(3) \quad \mu \geqq 1$$

$$(4) \quad \lambda - 2\mu \geqq -5$$

$$(5) \quad -\lambda + \mu \geqq 2.$$

This miniature linear program can readily be analyzed graphically. In a $\lambda$, $\mu$-coordinate plane (see Figure 1) we plot the "halfplanes" (1)–(5). In each case the halfplane (which includes its boundary line) is labelled by its number along its boundary line on the side of the line in which the halfplane lies: for example, the label (1) appears along the line $-\lambda + 2\mu = 2$ within the halfplane $-\lambda + 2\mu \geqq 2$. The set of *feasible* points $(\lambda, \mu)$ satisfying all five constraints is a quadrilateral determined by (2), (3), (4), (5). The first constraint is inactive, as it happens, because the feasible set lies entirely within the "open halfplane" $-\lambda + 2\mu > 2$. The

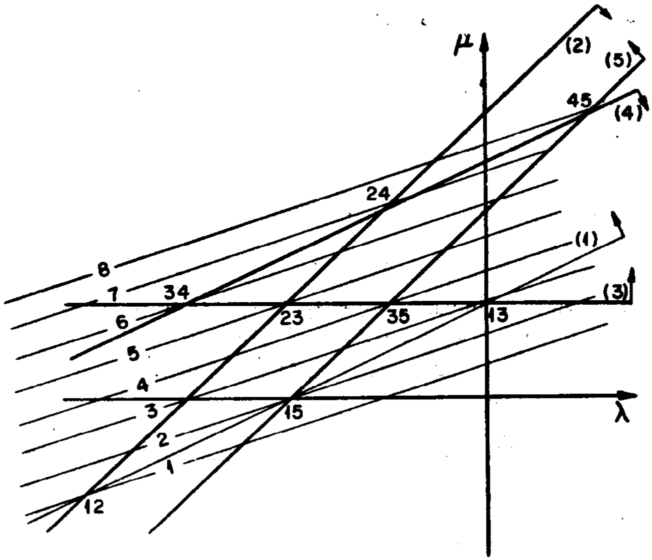


Fig. 1. Graphical analysis of the miniature linear program.

objective function, which is to be minimized, is represented by some particular contour (or level) lines along which it takes values as indicated. We see that the desired minimum appears to occur at the point of intersection of lines (3) and (5), namely at the point 35, with coordinates $\lambda = -1$, $\mu = 1$, where the objective function takes the value 4.

## SYSTEMATIC DISCUSSION OF EXAMPLE

The foregoing graphical method of solving a linear program does not generalize beyond two or three variables, whereas linear programs involving scores or hundreds of variables are not uncommon in actual practice. So, starting afresh, we outline a tabular systematization of our example which does generalize to large-scale problems.

To this end, we introduce the "tableau"

| | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $-1$ | |
|---|---|---|---|---|---|---|---|
| $\lambda$ | $-1$ | $1$ | $0$ | $1$ | $-1$ | $-1$ | $=0$ |
| $\mu$ | $2$ | $-1$ | $1$ | $-2$ | $1$ | $3$ | $=0$ |
| $-1$ | $2$ | $-3$ | $1$ | $-5$ | $2$ | $0$ | $=v$ |
| | $=x_1$ | $=x_2$ | $=x_3$ | $=x_4$ | $=x_5$ | $=u$ | |

consisting of a *matrix* of three rows and six columns inside the box and certain *marks* $\lambda$, $\mu$, $-1$, etc., around the four margins of the box. The six signs of equality along the bottom of the box indicate a "column system" of six linear equations

$$
\begin{aligned}
-\lambda + 2\mu - 2 &= x_1 \\
\lambda - \mu + 3 &= x_2 \\
\mu - 1 &= x_3 \\
\lambda - 2\mu + 5 &= x_4 \\
-\lambda + \mu - 2 &= x_5 \\
-\lambda + 3\mu &= u
\end{aligned}
$$

obtained by forming inner (or scalar) products of $\lambda$, $\mu$, $-1$ with the columns of the matrix and setting these inner products equal to $x_1$, $x_2$, $x_3$, $x_4$, $x_5$ and $u$. Then the constraints (1)–(5) of our example, as previously stated, become $x_1 \geqq 0$, $x_2 \geqq 0$, $x_3 \geqq 0$, $x_4 \geqq 0$, $x_5 \geqq 0$, and the objective function becomes $u$. We seek values of $\lambda$ and $\mu$ that minimize $u$ and make the five $x$'s nonnegative.

The tableau contains also a "row system" of three linear equations

$$\begin{aligned} -y_1 + y_2 \qquad + y_4 - y_5 + 1 &= 0 \\ 2y_1 - y_2 + y_3 - 2y_4 + y_5 - 3 &= 0 \\ 2y_1 - 3y_2 + y_3 - 5y_4 + 2y_5 \qquad &= v, \end{aligned}$$

indicated by the three signs of equality at the right-hand margin of the box. We obtain these equations by forming inner products of the rows of the matrix with $y_1, y_2, y_3, y_4, y_5, -1$ and setting these inner products equal to 0, 0, and $v$. This row system is essential to our analysis; it will give rise to a second linear program, "dual" to our first.

We observe, by substituting for the $x$'s and $u$ from the column system and then reducing via the row system, that

$$\begin{aligned} & x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 + x_5y_5 - u \\ = \ & \lambda(-y_1 + y_2 \qquad + y_4 - y_5 + 1) \\ & + \mu(2y_1 - y_2 + y_3 - 2y_4 + y_5 - 3) \\ & - (2y_1 - 3y_2 + y_3 - 5y_4 + 2y_5 \qquad) \\ = \ & \lambda(0) + \mu(0) - (v). \end{aligned}$$

Thus, the inner product of the marks at the bottom margin of the tableau with the marks at the top margin is equal to the inner product of the marks at the left margin with the marks at the right margin. By rearranging terms, we have

$$u - v = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 + x_5y_5$$

for any solutions of the column and row systems. This fundamental equation, in which $\lambda$ and $\mu$ do not appear explicitly, we call the "key equation" (of duality).

A solution $(\lambda, \mu; x_1, x_2, x_3, x_4, x_5, u)$ of the column system, or "column-solution," is *feasible* if its five $x$'s are nonnegative. Similarly, a solution $(y_1, y_2, y_3, y_4, y_5, v)$ of the row system, or "row-solution," is *feasible* if its five $y$'s are nonnegative. We see from the key equation that

$$u \geqq v$$