Fundamentals of Magnetohydrodynamics

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PREFACE

In this book we attempt to introduce the reader to the ideas of magnetohydrodynamics (MHD), to show him various methods which are used in MHD, and to prepare him for reading the original literature. These aims have determined our choice of material and the way we have presented it.

Often researchers aim at getting good agreement between experimental data and theoretical expositions, but this often is obtained at the expense of clarity and simplicity. The theory may be exact, but it is very complicated and often connected with computer solutions that describe only one particular solution of the equations and that are valid only for finite times. In other words, the theory may describe some experiments, but may not give any impulse to look for fundamentally new ideas—unexpected experiments or unexpected applications. Moreover, numerical solutions may not show ways for increasing the efficiency of MHD devices and the domains where they can be applied optimally. Finally, such approximate solutions cannot serve for finding the nature of the solution for infinite times.

This is the reason why researchers are sometimes driven to another extreme—crude qualitative reasoning is substituted for mathematical proof. However, the sad experience of *perpetuum mobile* inventors shows that qualitative arguments must be supplemented by a quantitative analysis.

We have chosen a "golden mean": a mathematical study of simplified models. Solutions obtained from simple models can serve as guides for approximate solutions of more complicated models. The unification of hydrodynamics and electromagnetism not only creates new difficulties, but it also leads to new physical phenomena. Solutions to simple problems can serve as a bridgehead for penetrating into the inner core of physical processes.

We restrict ourselves to a relatively small number of relatively straightforward main topics, but we shall work them out in detail. We shall use detailed calculations and analytical means to study the assumptions made and the methods applied. There are only a few cases where we consider for the sake of completeness of exposition problems which need cumbersome solutions and in those cases we refer the reader to the relevant literature.

We have tried to make the book easy to read and understand for readers who are not MHD specialists. Unfortunately, we did not have space for examples to illustrate the theory and, especially, for applications. Our main focus is on the concepts of MHD and on the physical picture of MHD phenomena without cluttering them with details. On the other hand, we explain in some detail various theoretical difficulties.

The subject of this book touches upon a great many important disciplines—electrodynamics, hydrodynamics, plasma physics, thermodynamics, aerodynamics, astrophysics, rheology, as well as such applied disciplines as electrical engineering, chemical technology, thermonuclear fusion, and jet technology. This has meant that we have had to incorporate some elementary discussions of some basic topics. Specialists in those subjects may find these discussions superfluous, but we found them necessary for specialists in other fields and for students.

Our book is mainly theoretical in character, but because it is short and simple, we feel that it comprises a "theoretical minimum" in MHD for experimental physicists and research engineers. We hope that the book will be used by lecturers and students of technical institutes and universities. It may also be useful for mathematicians who are interested in applications and—because of its extensive subject index—as a reference book.

A characteristic feature of the book is the width of the scope of methods and results of modern MHD and the exposition of its main ideas. In that respect it differs from many textbooks and survey articles which are devoted to particular applications. We may mention as such the books by Jeffrey and Taniuti,[119]† Anderson,[128] Bateman,[73] and Moffatt,[12] or the papers by Freidberg,[48] Liberman and Velikovich,[129] and Sagdeev.[125] The books by Pai,[162] Sutton and Sherman,[23] and Shercliff[163] give a wide coverage, but are now out-of-date as they do not cover recent developments, such as nonlinear waves, stability of equilibrium configurations, magnetoaerodynamics, and nonclassical MHD, which includes non-Newtonian fluids, magnetoelasticity, magnetoplasticity, ferrohydrodynamics, or electrohydrodynamics. The same is, of course, true of the pioneering books by Alfvén and Fälthammer[3] and by Kulikovskii and Lyubimov.[9]

The closest to the present book is the relevant chapter in Landau and Lifshitz's "Electrodynamics of Continuous Media," but because it is only one chapter many topics are of necessity not covered there. In particular, we may mention nonlinear waves, stability problems, shock wave structure, energy release or absorption, magnetoaerodynamics, and nonclassical MHD.

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† The numbers refer to the reference section at the end of the book.

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PART I

ELECTRICALLY CONDUCTING VISCOUS FLUIDS

CHAPTER 1

Basic Equations

1.1. Magnetohydrodynamic Approximations

When a conducting fluid or an ionized gas—a plasma—moves in a magnetic field an electric field is produced and an electric current appears. In turn, the interaction of the current with the magnetic field changes the motion of the fluid and changes the magnetic field.

Magnetohydrodynamics is that part of the mechanics of continuous media which studies the motion of electrically conducting media in the presence of a magnetic field.[63] In other words, magnetohydrodynamics studies the physics of fluid—or gaseous—conductors in a magnetic field. Apart from this definition, in which the *subject* of the studies is indicated, but not the *approximation* which is applied (magnetohydrodynamics in the **wider sense**), one understands by magnetohydrodynamics the *low-frequency limit* when one neglects not only kinetic effects, which occur due to the thermal spread of the particles, but also the difference in motion of the various components of the plasma—the electrons, various kinds of ions, and neutral particles. In magnetohydrodynamics in the **narrower sense**—the MHD model—one considers the medium as a single fluid which in each point of space $\mathbf{r}(x, y, z)$ and at each time t has a well-defined density ρ, pressure p, and velocity $\mathbf{v}$. Apart from these purely hydrodynamic quantities, the state of the medium is characterized by the magnetic induction $\mathbf{B}(\mathbf{r}, t)$. We assume that the magnetic permittivity μ is equal to unity and we do not distinguish between the magnetic induction $\mathbf{B}$ and the magnetic field strength $\mathbf{H}$.

We can use a hydrodynamic description if the frequency ω which is characteristic for the process we consider is appreciably smaller than the collision freqency ν of the separate particles: $\omega \ll \nu$. If we introduce a characteristic length L—size of the object—and a characteristic time τ—the time it takes ions with a thermal velocity $v_{T\mathrm{i}}$ to traverse a length L,

$$\tau \sim \frac{L}{v_{T\mathrm{i}}},$$

we have a characteristic frequency

$$\omega \sim \frac{1}{\tau} \sim \frac{v_{T\mathrm{i}}}{L},$$

and the condition $\omega \ll \nu$ takes the form $l \ll L$, where l is the mean free path.

Introducing the *Knudsen number* Kn,

$$\mathrm{Kn} = \frac{l}{L}, \tag{1.1}$$

we can say that the hydrodynamic description is valid for small Knudsen numbers, $\mathrm{Kn} \ll 1$. In this case, each kind of particles α—electrons, ions, neutrals—is described by their own density ρ_α, pressure p_α, and velocity $\mathbf{v}_\alpha$. This is *many-fluid magnetohydrodynamics*. In particular, a fully ionized plasma, consisting only of electrons and a single kind of ions, corresponds to *two-fluid magnetohydrodynamics*.

If the characteristic frequency ω is appreciably smaller than the frequency ν_{ei} for the exchange of energy between electrons and ions, the electron pressure p_{e} and the ion pressure p_{i} manage to become equal and the plasma is characterized by a single pressure p. Since the electrons are much lighter than the ions, the mass density ρ and the velocity $\mathbf{v}$ of the medium are then determined by the ions, and the current density $\mathbf{j}$ by the electrons. Such a fluid, considered as a single entity, is called a *simple* fluid.

Magnetohydrodynamics in the narrower sense is thus the low-frequency limit of magnetohydrodynamics in the wider sense. The importance of the low-frequency limit is due to the fact that low ferrequencies correspond to long wavelengths. Therefore, large amounts of matter are involved in the low-frequency motions. As to high-frequency, that is, short-wavelength, processes, small-scale details of motion correspond to them, like ripples on the water surface. Moreover, the interference of different frequencies, that is, different wavelengths, contained in a wavepacket leads to damping of the short-wavelength components. This effect is well known from acoustics—far from an orchestra only the drum is heard. Therefore, over long time intervals and at large distances from the source the long-wavelength terms of the relevant quantities play the main role.

In what follows, we mean by magnetohydrodynamics magnetohydrodynamics in the *narrower sense.*

For slow motions of the medium the electrons are displaced in the direction in which the electrical potential increases in such a way that the gradient of this potential becomes zero. The electrical field $\mathbf{E}'$ in the eigenframe,[2]†

$$\mathbf{E}' = \mathbf{E} + \frac{1}{c}\,[\mathbf{v} \times \mathbf{B}], \tag{1.2}$$

is then equal to zero, that is,

$$\mathbf{E} = -\frac{1}{c}\,[\mathbf{v} \times \mathbf{B}] \tag{1.3}$$

(we assume that the plasma velocity is nonrelativistic, $v \ll c$, and we therefore neglect terms of order v^2/c^2).

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† We neglect the effects of the polarization and magnetization of the medium; we thus put the electric induction[1] $\mathbf{D}$ equal to the electric field strength $\mathbf{E}$ and we put the magnetic induction $\mathbf{B}$ equal to the magnetic field strength $\mathbf{H}$. In other words, we assume that the electric permittivity ϵ and the magnetic permittivity μ are equal to unity.

It is clear from (1.3) that the electric field in nonrelativistic magnetohydrodynamics is appreciably smaller than the magnetic field:

$$E \sim \frac{v}{c} B. \tag{1.4}$$

The magnetic field therefore is independent of the frame of reference.

It is clear from (1.4) that the energy $B^2/8\pi$ of the magnetic field in nonrelativistic magnetohydrodynamics is much larger than the energy $E^2/8\pi$ of the electrical field and that we can neglect the latter.

The single-fluid hydrodynamic approximation describes a large class of phenomena in *nonionized gases* very well. In contrast, single-fluid *magneto*hydrodynamics describes the experimental data satisfactorily only on cosmic scales. In the case of a laboratory plasma single-fluid magnetohydrodynamics is a rather coarse approximation to reality. As far as conducting fluids—liquid metals, sea water—under terrestrial conditions are concerned, they are well described in the framework of a single-fluid model.

In magnetohydrodynamics there are unusual relations between the various electrical quantities. For instance, in *electrical engineering* the current is basically determined by the electric field—Ohm's law—and the magnetic field causes only minor corrections. On the other hand, in *magnetohydrodynamics* the current is determined mainly by the magnetic field.

Indeed, let us estimate the different terms in the Maxwell equation

$$\operatorname{curl} \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j}. \tag{1.5}$$

Using Eq.(1.4) we get

$$|\operatorname{curl} \mathbf{B}| \sim \frac{B}{L}, \qquad \frac{1}{c}\left|\frac{\partial \mathbf{E}}{\partial t}\right| \sim \frac{v}{c^2\tau} B,$$

where L is a characteristic length and τ a characteristic time. Noting that the ratio L/τ is equal to a characteristic velocity V and that v is also of order V, we find that

$$\frac{1}{c}\left|\frac{\partial \mathbf{E}}{\partial t}\right| \Big/ |\operatorname{curl} \mathbf{B}| \sim \frac{V^2}{c^2}.$$

Hence, in the nonrelativistic case, $V \ll c$, the displacement current on the right-hand side of Eq. (1.5) can be neglected. Equation (1.5) thus enables us to express the current density in terms of the magnetic induction:

$$\mathbf{j} = \frac{c}{4\pi} \operatorname{curl} \mathbf{B}. \tag{1.6}$$

If the characteristic velocity is of the order of or larger than the sound velocity, we must take the compressibility of the medium into account, that is, assume the density ρ to be variable. To emphasize this fact one sometimes uses instead of the term "magneto*hydrodynamics*" the term "magneto*gasdynamics*." Sometimes the terms *magnetohydrodynamical* and *magnetogasdynamical* replace the equivalent terms **hydromagnetic** and **gasmagnetic**. We shall call a magnetohydrodynamical medium a "fluid" or a "plasma" depending on the applications and the traditional nomenclature.

We note that the magnetohydrodynamics approximation of an *incompressible* fluid is sometimes used outside the limits of its applicability. This is connected with the fact that taking the compressibility into account considerably complicates the MHD effects whereas these effects show up more clearly in an incompressible fluid.

In thermonuclear studies the magnetohydrodynamical equations were at the end of the fifties used as a first approximation. Later the center of attention was shifted to the more exact, but more complex, kinetic theory. However, the kinetic description contains much more information. It is impossible to solve the set of kinetic equations, not even numerically in the case of a nontrivial geometry.

In the seventies interest shifted again to magnetohydrodynamics in connection with the transition in thermonuclear studies to denser plasmas and more complex methods of containment.

1.2. Magnetohydrodynamical Effects

A number of physical conclusions and technical applications follow from the equations of § 1.1. It is, first of all, clear from Eq. (1.3) that the motion of a conducting medium across a magnetic field produces a difference in electric potentials. This effect can be used to produce *magnetohydrodynamical generators*[4,5] for electric energy in which a direct transition from thermal to electric energy occurs. In order that a gas becomes electrically conducting, it is heated to some thousands of degrees, which produces ionization, that is, changes the gas into a plasma. The main advantage of MHD generators over thermal ones—such as gas turbines—consists in the fact that the plasma has a high temperature T and this leads to an increase in the maximum efficiency which is known to equal[6]

$$\eta_{\max} = 1 - \frac{T_0}{T},$$

where T_0 is the temperature of the surrounding medium and T the temperature of the working body. For increasing $\eta_{\max}$ we need high temperatures T, which cannot be reached in gas turbines since at high temperatures their blades lose strength and break.

Other applications of Eq. (1.3) include *magnetohydrodynamic flow meters*[7] and *velocity gauges*. The advantage of an MHD flow meter is its small inertia.

It also follows from (1.3) that a plasma placed in crossed electric and magnetic fields is set in motion. If, for instance, the electric field is along the y-axis and the magnetic field along the z-axis, there is a component of the velocity of the medium along the x-axis:

$$v_x = \frac{cE_y}{B_z}. \tag{1.7}$$

Such a motion, which is perpendicular to both the electric and the magnetic fields, is called *drift* and its velocity is called the *drift velocity*. This effect is used to produce *magnetohydrodynamic engines* for rockets.[8] The plasma can in this case be given a velocity which is much larger than the sound velocity with which the gases are ejected from the nozzle of the usual rocket. The thrust developed per unit mass of fuel in a magnetohydrodynamic reactive engine is thus much larger than in a thermal engine.

The motion of an electrically conducting fluid in crossed electric and magnetic fields is also used to produce *conduction type magnetohydrodynamic pumps*; the motion of the fluid is accompanied by a conduction current of density j_y which is directed across the pipe and caused by the electrical field E_y. The advantage of MHD pumps