NUMERICAL METHODS IN FINITE ELEMENT ANALYSIS

KLAUS-JÜRGEN BATHE

EDWARD L. WILSON

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KLAUS-JÜRGEN BATHE
Department of Mechanical Engineering
Massachusetts Institute of Technology

EDWARD L. WILSON
Department of Civil Engineering
University of California, Berkeley

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PREFACES

During the last years, the finite element method of analysis has rapidly become a very popular technique for the computer solution of complex problems in engineering. Basically, the method can be understood as an extension of earlier established analysis techniques, in which a structure is represented as an assemblage of discrete truss and beam elements. The same matrix algebra procedures are used, but instead of truss and beam members, finite elements are employed to represent regions of plane stress, plane strain, axisymmetric, three-dimensional, plate, or shell behavior.

In the earliest developments of finite element analysis, quite naturally almost all emphasis was directed toward the development of effective finite elements for the solution of specific problems. However, the potential of the method when used effectively on a digital computer was rapidly realized, and increasingly larger and more complex finite element systems were considered. This, in turn, enhanced the development of efficient data-handling procedures and effective techniques for the solution of the governing finite element equilibrium equations. At present, computer programs are in use that can handle at reasonable cost very large finite element systems, because the algorithms employed have been developed specifically for finite element analysis. As a result of this development, when referring to finite element analysis, a complete numerical process implemented on the digital computer is implied. This numerical process comprises the formulation of the finite element matrices, the numerical integration to evaluate the matrices, the assemblage of the element matrices into the matrices that correspond to the complete finite element system, and the numerical solution of the system equilibrium equations.

The objective in this book is to present each of the above aspects of finite element analysis and thus to provide a basis for the understanding of the complete solution process. Because finite element analysis is basically a numerical procedure, the numerical aspects of the method are emphasized, but whenever possible physical explanations are given.

According to three basic areas in which knowledge is required, the book is divided into three parts. In the first part, important concepts of matrix and linear algebra are presented. Many readers may be familiar with the elementary rules of matrix algebra, but specific attention should be given to the concepts of linear algebra, because they provide the foundation for a thorough understanding of the numerical procedures presented later.

The second part of the book comprises the formulation of the finite element method and the numerical procedures used to evaluate the element matrices and the matrices of the complete element assemblage. Since the early use of finite element analysis, numerous finite elements have been developed. The objective in this book is not to summarize all the finite element models available, but rather to establish the general principles and describe those elements that are believed to be most effective at present.

In the last part of the book, procedures for the effective solution of the finite element equilibrium equations in static and dynamic analysis are presented. This phase of finite element analysis usually comprises most of the computer effort and, consequently, deserves a great deal of attention. It should be realized that because the numerical procedures employed in this phase largely determine the cost of analysis, they also determine whether an analysis can or cannot be performed in practice.

Throughout the presentation, the aim was to establish the numerical procedures using relatively simple and physical concepts, even at the expense of losing some mathematical rigorousness. The principles and procedures are illustrated using well over 100 worked-out examples, which are an integral part of the presentation. Short computer programs are also included to demonstrate the numerical procedures in a compact manner. These programs can be used directly as subroutines in finite element codes.

The primary aim in the writing of this book is to provide a tool for teaching upper-level undergraduate and graduate courses in engineering. Although the topic is finite element analysis procedures, many of the numerical techniques are quite general and could be employed effectively in any discrete method of analysis. In addition, a large community of engineers using finite element computer programs should find much valuable information in the text.

A very difficult aspect of writing a book is to give references that appropriately acknowledge the work of the various researchers in the field. This is particularly difficult in this book, because the field of finite element analysis has been expanding very rapidly. I would like to apologize here for possibly not having referred at all times to the most appropriate published work.

The endeavor to write this book is a result of the excitement and challenge that I have experienced in working in the field of finite element analysis. I would like to thank my former teacher, Edward L. Wilson, who through his active support made it possible for me to pursue my research and complete most of the work on this book while at Berkeley. Although I have been writing this book and the responsibility for the book rests with me, the name of E. L. Wilson on the title page is justified by the fact that my research and development work was largely based on his earlier achievements.

For some years, I have been associated with Fred Peterson of Engineering/Analysis Corporation, Berkeley, whom I would like to thank for his support. I much enjoyed working with Fred. I also would like to thank Steve Kenney for helping me on the problem solutions; Aileen Frankel, who did an outstanding job in typing the manuscript; Bill Grace for helping me during the final phase of work; and Rosalie Herion, the editor of the book, for her patience and diligence. Finally, I would like to thank, in particular, my wife, Zorka, who with her love and patience supported the writing of this book.

K.J. BATHE
M.I.T.

Researchers in both engineering mechanics and applied mathematics have participated in the development of the finite element method. During the period 1850 to 1860, theories on torsion and bending of beams were unified, and the foundations of the field of structural analysis were established. For approximately 100 years, structural analysis was restricted to the study of systems of one-dimensional beam and truss elements, an important characteristic being that an element was always only connected to two joints. In the mid-1950s, two-dimensional structural elements connected to more than two joints were developed in the aircraft industry in order to improve the modeling of the stiffness of thin membrane elements connected to traditional one-dimensional structural elements. In 1960, Clough first introduced the finite element terminology in his paper, "The Finite Element Method in Plane Stress Analysis." In this paper, the method was presented as an extension of structural analysis techniques to the solution of problems in continuum mechanics.

Ritz, in 1909, developed a very powerful method for the approximate solution of field problems in continuum mechanics. This approach involves the approximation of a potential functional in terms of trial functions of unknown magnitudes. The minimization of the functional with respect to each unknown results in a set of equations that are solved for the unknown magnitudes. One of the initial limitations of the Ritz method was that the trial functions satisfy the boundary conditions of the problem. Courant, in 1943, made a significant extension of the Ritz method by introducing separate linear functions over triangular areas and applied the method to the solution of torsion problems. The unknowns in the problem were selected as the values of the functions at the locations of the interconnecting triangular areas. Hence, the traditional limitation of the global Ritz functions in satisfying boundary conditions was eliminated, since the conditions could be satisfied at a finite number of points along a boundary. The Ritz method as used by Courant was identical to the finite element method independently presented by Clough many years later. Of course, the reason that the finite element method met with almost immediate success in 1960 lies in the fact that the large number of numerical operations inherent in the method could be performed by the recently developed digital computer; whereas, this tool was not available to Courant in 1943.

In the mid-1960s, researchers in both the fields of continuum mechanics and structural analysis recognized that the extended Ritz and the finite element methods are identical, and within the next ten years, the development and application of the method progressed at a very impressive rate. The finite element method has been applied to problems in three dimensions, problems involving both material and geometric nonlinearities, time-dependent problems, and problems in many areas beyond structural analysis, such as fluid flow, heat transfer, and magnetic field analysis. Further comments on the history of the development of the method are presented in Chapter 3.

Because of the speed with which the finite element method has expanded, a number of books have recently been published. Most of these books are concerned with the development of various elements and with examples of their applications. The major purpose of this book is to present the fundamental numerical methods required in all finite element applications. In addition, emphasis is placed on the computer implementation of the numerical procedures. Therefore, I feel that this work is unique in the field when compared to other finite element books.

The successful completion of research involving finite element development generally results in the development of a computer program to be used in the practical solution of engineering problems. In order for the program to be an effective analysis tool, it must be based on theories and techniques from three different disciplines. First, the approximations used to develop the properties of the various finite elements must be based on sound fundamental principles of continuum mechanics. Second, the numerical methods selected for spacial integration, solution of equations, evaluation of eigenvalues, and step-by-step time solutions must be accurate and efficient. Third, the computer implementation of the numerical techniques used must be approached with great care if the number of numerical operations are to be minimized, high speed and low speed storage units are to be used effectively, and the resulting program is to be reasonably machine-independent. The purpose of this book is to present the necessary background in all three of these areas.

The development of new finite elements, effective numerical methods, and practical computer programs have been the areas of my personal research of the past 15 years, and this book emphasizes these research areas. During the past five years, my research in these areas has been greatly enhanced due to my collaboration with Klaus-Jürgen Bathe. Working with Jürgen has been a most enjoyable experience.

E. L. WILSON
U. C., Berkeley

CONTENTS

PART I

MATRICES AND LINEAR ALGEBRA

CHAPTER 1

ELEMENTARY CONCEPTS OF MATRICES

1.1 INTRODUCTION

The practical use of finite element analysis is based on matrix algebra and the use of the electronic computer, because it is only in matrix form that the complete solution process can be expressed in a compact and elegant manner. The objective in this chapter is to present briefly the fundamentals of matrix algebra that are needed for an understanding of the solution procedures discussed later.[1-3] In the presentation, emphasis is directed to those aspects of matrix algebra that are important in finite element analysis.

From the practical point of view, matrix algebra can be regarded merely as an effective tool with which to manipulate in an elegant manner large amounts of data, and we use this approach in this chapter. Some concepts of linear algebra that are also required for a thorough understanding of the finite element method are presented in Chapter 2.

Consider that the objective of finite element analysis is to evaluate the displacements at a large number of points of the structure under consideration. Then, once the physical relationships between the required displacements of the structure and the applied loads are known, it is possible, using the concepts of matrix algebra, to express the complete solution process employing a few symbols to identify the various quantities and a few lines to describe the solution process. However, although the solution process is expressed in an elegant manner in terms of matrices and matrix manipulations, it is frequently important to identify the detailed operations that are followed in the matrix solution in order to design more effective solution