Exact Solutions of Relativistic Wave Equations

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SERIES EDITOR'S PREFACE

'Et moi, ..., si j'avait su comment en revenir, je n'y serais point allé.'

Jules Verne

The series is divergent; therefore we may be able to do something with it.

O. Heaviside

One service mathematics has rendered the human race. It has put common sense back where it belongs, on the topmost shelf next to the dusty canister labelled 'discarded non-sense'.

Eric T. Bell

Mathematics is a tool for thought. A highly necessary tool in a world where both feedback and non-linearities abound. Similarly, all kinds of parts of mathematics serve as tools for other parts and for other sciences.

Applying a simple rewriting rule to the quote on the right above one finds such statements as: 'One service topology has rendered mathematical physics ...'; 'One service logic has rendered computer science ...'; 'One service category theory has rendered mathematics ...'. All arguably true. And all statements obtainable this way form part of the raison d'être of this series.

This series, *Mathematics and Its Applications*, started in 1977. Now that over one hundred volumes have appeared it seems opportune to reexamine its scope. At the time I wrote

> "Growing specialization and diversification have brought a host of monographs and textbooks on increasingly specialized topics. However, the 'tree' of knowledge of mathematics and related fields does not grow only by putting forth new branches. It also happens, quite often in fact, that branches which were thought to be completely disparate are suddenly seen to be related. Further, the kind and level of sophistication of mathematics applied in various sciences has changed drastically in recent years: measure theory is used (non-trivially) in regional and theoretical economics; algebraic geometry interacts with physics; the Minkowsky lemma, coding theory and the structure of water meet one another in packing and covering theory; quantum fields, crystal defects and mathematical programming profit from homotopy theory; Lie algebras are relevant to filtering; and prediction and electrical engineering can use Stein spaces. And in addition to this there are such new emerging subdisciplines as 'experimental mathematics', 'CFD', 'completely integrable systems', 'chaos, synergetics and large-scale order', which are almost impossible to fit into the existing classification schemes. They draw upon widely different sections of mathematics."

By and large, all this still applies today. It is still true that at first sight mathematics seems rather fragmented and that to find, see, and exploit the deeper underlying interrelations more effort is needed and so are books that can help mathematicians and scientists do so. Accordingly MIA will continue to try to make such books available.

If anything, the description I gave in 1977 is now an understatement. To the examples of interaction areas one should add string theory where Riemann surfaces, algebraic geometry, modular functions, knots, quantum field theory, Kac-Moody algebras, monstrous moonshine (and more) all come together. And to the examples of things which can be usefully applied let me add the topic 'finite geometry'; a combination of words which sounds like it might not even exist, let alone be applicable. And yet it is being applied: to statistics via designs, to radar/sonar detection arrays (via finite projective planes), and to bus connections of VLSI chips (via difference sets). There seems to be no part of (so-called pure) mathematics that is not in immediate danger of being applied. And, accordingly, the applied mathematician needs to be aware of much more. Besides analysis and numerics, the traditional workhorses, he may need all kinds of combinatorics, algebra, probability, and so on.

In addition, the applied scientist needs to cope increasingly with the nonlinear world and the extra mathematical sophistication that this requires. For that is where the rewards are. Linear models are honest and a bit sad and depressing: proportional efforts and results. It is in the non-linear world that infinitesimal inputs may result in macroscopic outputs (or vice versa). To appreciate what I am hinting at: if electronics were linear we would have no fun with transistors and computers; we would have no TV; in fact you would not be reading these lines.

There is also no safety in ignoring such outlandish things as nonstandard analysis, superspace and anticommuting integration, p-adic and ultrametric space. All three have applications in both electrical engineering and physics. Once, complex numbers were equally outlandish, but they frequently proved the shortest path between 'real' results. Similarly, the first two topics named have already provided a number of 'wormhole' paths. There is no telling where all this is leading - fortunately.

Thus the original scope of the series, which for various (sound) reasons now comprises five subseries: white (Japan), yellow (China), red (USSR), blue (Eastern Europe), and green (everything else), still applies. It has been enlarged a bit to include books treating of the tools from one subdiscipline which are used in others. Thus the series still aims at books dealing with:

- a central concept which plays an important role in several different mathematical and/or scientific specialization areas;
- new applications of the results and ideas from one area of scientific endeavour into another;
- influences which the results, problems and concepts of one field of enquiry have, and have had, on the development of another.

The Klein-Gordon and the Dirac equations are the basic equations of relativistic quantum mechanics and they are of fundamental importance. Until the early seventies only about a dozen external magnetic fields were known for which exact solutions could be written down. Now there are hundreds, thanks mainly to significant advances in the theory of separation of variables and the closely related theory of symmetries of partial differential equations.

This unique book systematically treats all these solutions and the general theory behind them.

The shortest path between two truths in the real domain passes through the complex domain.

J. Hadamard

La physique ne nous donne pas seulement l'occasion de résoudre des problèmes ... elle nous fait pressentir la solution.

H. Poincaré

Never lend books, for no one ever returns them; the only books I have in my library are books that other folk have lent me.

Anatole France

The function of an expert is not to be more right than other people, but to be wrong for more sophisticated reasons.

David Butler

Bussum, January 1990

Michiel Hazewinkel

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Introduction

The relativistic wave equations of Dirac and Klein-Gordon describing the motion of an electric charge in an external electromagnetic field provide a basis for relativistic quantum mechanics and quantum electrodynamics of spinor and scalar particles. There are many reasons why exact solutions of these equations are of special physical interest. In relativistic quantum mechanics, the Dirac and Klein-Gordon equations are referred to as one-particle wave equations of motion for fermions and bosons in an external electromagnetic field: their solutions describing the motion [1]. In quantum electrodynamics, exact solutions of these equations make it possible to develop the perturbation expansion known as the Furry picture which incorporates the interaction with the external field exactly, while treating the interaction with the quantized photon field perturbatively [2 - 4]. In particular, all propagators of a particle, i.e., the various Green's functions, are constructed in a certain way by using exact solutions of the Dirac and Klein-Gordon equations.

One of the basic equations of relativistic quantum theory, the Klein-Gordon equation, was known already to Schrödinger [5]. The final form of the scalar relativistic-covariant wave equation was independently established in [6 - 10], in which its properties were also discussed. These works appeared almost simultaneously and independently. For this reason, it is impossible to allocate a sufficiently short name to this equation which would do justice to all of its authors. The most widespread, although not the most historically adequate, is the name "Klein-Gordon equation."

The equation which most exactly describes the motion of electrons was given by Dirac [11, 12] and bears his name.

The physically most important exact solutions of these equations were obtained and analysed in the early years of relativistic quantum theory.

These are the solution for an electron in a Coulomb field [5, 11, 13 - 15], a uniform magnetic field [16 - 20], the field of a plane wave [21, 22] and some simple one-dimensional electric fields [20, 23, 24].

During the subsequent three decades, only one solution of the Dirac and Klein-Gordon equations was found, namely, that for a charge placed in the field of a magnetic monopole [25]. Beginning in the mid-sixties however, new works appeared related to the search for new exact solutions. Such solutions were found: for an electron in the field of a plane wave combined with a uniform magnetic field parallel to the direction of wave propagation (the Redmond field) [26]; for an electron in some non-uniform fields [27 - 33]; the stationary solution in constant and homogeneous electric and magnetic fields that are equal in magnitude and mutually orthogonal (crossed fields) [34 - 37]; and in the field of a wave with an isotropic 4-potential [38].

In the early seventies about a dozen external electromagnetic fields were listed for which exact solutions of the Dirac and Klein-Gordon equations had been determined. However, no general method for finding such solutions was available.

This situation changed significantly after works appeared in which the problem of listing all external fields that allow a complete separation of variables in the Dirac [39, 40] and Klein-Gordon [41, 42] equations was solved. These investigations were possible following an essential advance in the theory of separation of variables in second-order differential equations and systems of first-order differential equations [39 - 46].

Solving the Dirac or Klein-Gordon equation means either solving the Cauchy problem, i.e., finding the wave function at any instant in time using the data given at an initial time, or determining the complete system of solutions that are, simultaneously, eigenfunctions of a complete set of operator-valued integrals of motion. Most of the solutions known belong to the latter type.

It is known that for the Klein-Gordon equation the complete set contains three operator-valued integrals of motion. These three integrals of motion usually have direct classical analogues which makes the parallel solution of the corresponding classical problem especially interesting. For the Dirac equation, the complete set contains four integrals of motion, with at least one not admitting the classical interpretation (the so-called spin operator-valued integral of motion). All operators of the complete set should commute among themselves and with the operator of the equation and be functionally independent. The method of separation of variables contains in itself the finding of complete sets of integrals of motion (to be more precise, of complete sets of symmetry operators[1] that are also integrals of motion). It should be mentioned that all known exact solutions of the Dirac and Klein-Gordon equations were obtained for fields belonging to the class found in [41].

The explicit determination of all fields allowing separation of variables opened wide possibilities for finding new exact solutions of the Dirac and Klein-Gordon equations. At present, hundreds of such solutions are known. A need has arisen to review, at least briefly, the known solutions, to emphasize their common features, and to distinguish their peculiarities. This is the task we try to fulfill in the present book.

The second and third chapters of the book collect the now known exact solutions of the Dirac and Klein-Gordon equations which can be expressed as combinations of a finite number of elementary and higher transcendental functions. We have chosen to classify the solutions according to the type of external fields involved. This means the following For many external fields (e.g. the plane-wave or homogeneous magnetic field) different complete systems of solutions are known, depending on different choices of the sets of integrals of motion. For the homogeneous magnetic field, for example, there are various complete systems of solutions, widely used in the literature and presented also in this book. They derive from separation of variables performed in Cartesian and cylindrical reference frames. There are also the coherent states and some other more special complete systems. However, it is not our intention to list for every given field all the complete systems of solutions ever described in the literature. Our purpose has been to indicate all the external electromagnetic fields allowing exact solutions of the Dirac or Klein-Gordon equations and to present at least one complete system of solutions for each such field. The physically most important and most often used solutions are presented in Chapter 2.

It is quite clear that once an exact solution of a relativistic-covariant

[1] A symmetry operator of a given equation is an operator that maps every solution of this equation into a solution of the same equation. Details may be found in [40 - 46].

equation of motion is known for a certain external field, its solution is therefore known for the whole class of fields obtained from the given field by Lorentz transformations. In every case, we have chosen the simplest representative from a given class of fields. We have done our best to cite original papers wherever possible, although we have not always mentioned complementary results which have appeared later.

It should be noted that, in general, solving the Dirac equation does not always reduce to solving the Klein-Gordon equation for the same electromagnetic field. (The class of fields for which a connection between solutions of these equations is established is described in section 3 of Chapter 1.)

Nevertheless, in most cases, whenever an explicit solution of the Klein-Gordon equation is known, one succeeds in finding the explicit solution of the Dirac equation, too. This observation is not only confirmed in the three known cases [47], described later (§§ 37 - 39). The following statement is also true: if the Klein-Gordon equation allows, for a certain electromagnetic field, complete separation of variables, then the solution of the classical Lorentz equations can be reduced to performing quadratures; the classical action can be found as a quadrature, as well. In view of these facts, for every class of electromagnetic fields considered, we present, in succession, solutions of the classical Lorentz and Hamilton-Jacobi equations, and then of the Klein-Gordon and Dirac equations. As far as the latter two are concerned, we present their complete systems of solutions in Chapters 2 and 3. Exact solutions of the classical problem are of independent value; they are also useful in quantum theory, e.g. for the interpretation of integrals of motion.

The reader who may only be interested in the solution for a special electromagnetic field is advised to acquaint himself not only with the text that relates directly to this solution, but also to consult the beginning of the corresponding chapter and section since, as a rule, in that place notations that are to be used specifically for a given chapter and section are introduced, and other information that belongs to the case under consideration is given.

Apart from the division of the book into numbered chapters and sections, that part of the text relating to finding exact solutions is indicated by specific, separate texts: each corresponding to a special solution. These texts are numbered throughout the book as § 1, § 2, etc., and are also indicated in the Table of Contents.

Within each of these specific texts the solution of the Klein-Gordon and Dirac equations is usually reduced to solving a one-dimensional stationary Schrödinger equation with a certain potential. Therefore, in order to avoid repetition, all the known solutions of the one-dimensional stationary Schrödinger equation and all the potentials that allow solutions of this equation are listed at the end of the book in Appendix 1, and we refer to this Appendix when presenting the final results.

The contents of Chapters 4 - 6 relate closely to the main text concerning exact solutions of relativistic wave equations in external electromagnetic fields. In Chapter 4, exact solutions of these equations for the Green's functions are given. Special Green's functions are presented for the physically most interesting case of the constant electromagnetic field combined with the plane-wave field. In Chapter 5, we consider exact coherent state solutions of the Dirac and Klein-Gordon equations in an electromagnetic field. The fruitfulness of the use of coherent states in, e.g., electrodynamics and optics, is well-known; the Green's functions in an external field, as pointed out above, are the main building blocks of the perturbation expansion with respect to the radiative interaction in quantum electrodynamics. (Detailed reviews of the literature relating to Chapter 4 and the subsequent two chapters can be found in the introductions to these chapters.)

Chapter 6 is concerned with exact solutions of the Dirac and Klein-Gordon equations with an operator-valued electromagnetic potential, which represents the plane-wave quantized electromagnetic field. From the point of view of quantum field theory this equation describes the model problem of the interaction of a charge with the quantized electromagnetic field of photons having collinear momenta. Consideration of this problem is interesting because few exactly solvable models in quantum field theory exist. Although the models under consideration do not directly relate to any physical reality, their exact solution advances our understanding of the mathematical structure of the theory and serves to illustrate general ideas. For instance, in the classical limit with respect to the electromagnetic field, the solutions of the problem discussed turn into solutions of the Dirac and Klein-Gordon equations in an external plane-wave field – the Volkov solutions – but are more general than the latter since they incorporate, albeit in a model way, the backward influence of the motion of the charge on the plane-wave field itself. These solutions give, moreover, the possibility of calculating some actual many-photon effects of quantum electrodynamics.

In the last chapter, Chapter 7, we present exact solutions of the extended Dirac equation, which takes into account the fact that spinor particles have anomalous magnetic and electric moments. A consistent treatment of the anomalous magnetic moment of an electron is possible only within the scope of quantum electrodynamics (QED), whereas the anomalous electric moment requires for its description a quantum field model with parity nonconservation. Nonetheless, the extended Dirac equation under consideration may be given an approximate theoretical foundation within QED, as well as a phenomenological justification, and is used as the simplest approximation for describing the influence of the anomalous moments on the physical parameters of various processes. We also indicate here all the external electromagnetic fields that enable one to obtain solutions of the extended Dirac equation, and we present (of the many possible) one complete system of such solutions for each field.

It is hoped that this book will be of interest to specialists in the fields of quantum mechanics, field theory, and mathematical physics. We hope that the book will also provide a new stimulus to investigations of, and searches for, exact solutions of the relativistic wave equations and related equations of mathematical physics.

In conclusion, the authors wish to use this opportunity to express their gratitude to their colleagues and students who participated in discussions of the material presented.

Chapter 1
GENERAL PROPERTIES OF SOLUTIONS OF THE KLEIN-GORDON AND DIRAC EQUATIONS

1 Basic notations and equations

The Cartesian coordinates of space-time points are denoted by $x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z) = (x^0, \boldsymbol{x})$, where c is the speed of light. In this frame, the metric tensor is

$$\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1). \tag{1.1}$$

The summation convention over repeated sub- and super- scripts is assumed throughout, unless otherwise explicitly stated. Contravariant vectors are often represented in the form $a^\mu = (a^0, \boldsymbol{a})$, and

$$(ab) = a^\mu b_\mu = \eta_{\mu\nu} a^\mu b^\nu = a^0 b^0 - (\boldsymbol{a}\boldsymbol{b}). \tag{1.2}$$

The electromagnetic field tensor $F_{\mu\nu}$ and the field strengths $\boldsymbol{E}$ and $\boldsymbol{H}$ are expressed in terms of the electromagnetic field potentials A_μ in the standard way:

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, \\ \boldsymbol{E} &= -\nabla A_0 - \partial_0 \boldsymbol{A}, \\ \boldsymbol{H} &= [\nabla \boldsymbol{A}]. \end{aligned} \tag{1.3}$$

We set

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \tag{1.4}$$

throughout.

In what follows, the matrix $F(\boldsymbol{a},\boldsymbol{b})$ will be useful:

$$F(\boldsymbol{a},\boldsymbol{b}) = \begin{pmatrix} 0 & a^1 & a^2 & a^3 \\ -a^1 & 0 & -b^3 & b^2 \\ -a^2 & b^3 & 0 & -b^1 \\ -a^3 & -b^3 & b^1 & 0 \end{pmatrix}, \tag{1.5}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are three-dimensional vectors given by their Cartesian components. The Cartesian components of the tensors $F_{\mu\nu}$ and $F^{\mu\nu}$ can then be written as

$$F_{\mu\nu} = F(\boldsymbol{E},\boldsymbol{H}),\ F^{\mu\nu} = F(-\boldsymbol{E},\boldsymbol{H}), \tag{1.6}$$

where μ is the number of the row, and ν that of the column.

The matrix transpose of A will be denoted by A^T.

The (pseudo)tensor dual to a tensor $T^{\mu\nu}$ will be denoted by $T^{*\mu\nu}$,

$$T^{*\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}T_{\alpha\beta} \tag{1.7}$$

where $\epsilon^{\mu\nu\alpha\beta}$ is the completely antisymmetric unit pseudotensor, $\epsilon^{0123} = 1$.

From (1.7) and (1.6) one obtains for the dual pseudotensor of the electromagnetic field:

$$\begin{aligned} F^{*\mu\nu} &= -F(\boldsymbol{H},\boldsymbol{E}), \\ F^{*}_{\mu\nu} &= -F(\boldsymbol{H},-\boldsymbol{E}). \end{aligned} \tag{1.8}$$

The electromagnetic field invariants have the form

$$\begin{aligned} I_1 &= \frac{1}{2}F_{\mu\nu}F^{\mu\nu} = \boldsymbol{H}^2 - \boldsymbol{E}^2, \\ I_2 &= \frac{1}{4}F^{*}_{\mu\nu}F^{\mu\nu} = (\boldsymbol{E}\boldsymbol{H}). \end{aligned} \tag{1.9}$$

As noted above, we are going to concentrate on the search for exact solutions of the Lorentz, Klein-Gordon and Dirac equations. We will list these equations, as well as those of Maxwell and Hamilton-Jacobi, below. In doing so, we refer to the Gauss system of units, with e being the algebraic