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PREFACE

This monograph is a collective work. The names appearing on the front cover are those of the people who worked on every chapter. But the contributions of others were also very important:

C. Risito for Chapters I, II and IV,

K. Peiffer for III, IV, VI, IX

R. J. Ballieu for I and IX,

Dang Chau Phien for VI and IX,

J. L. Corne for VII and VIII.

The idea of writing this book originated in a seminar held at the University of Louvain during the academic year 1971-72. Two years later, a first draft was completed. However, it was unsatisfactory mainly because it was excessively abstract and lacked examples. It was then decided to write it again, taking advantage of some remarks of the students to whom it had been partly addressed. The actual text is this second version.

The subject matter is stability theory in the general setting of ordinary differential equations using what is known as Liapunov's direct or second method. We concentrate our efforts on this method, not because we underrate those which appear more powerful in some circumstances, but because it is important enough, along with its modern developments, to justify the writing of an up-to-date monograph. Also excellent books exist concerning the other methods, as for example R. Bellman [1953] and W. A. Coppel [1965].

Liapunov's second method has the undeserved reputation of being mainly of theoretical interest, because auxiliary functions appear to be so difficult to construct. We feel this is the opinion of those people who have not really tried. Indeed, many mathematicians have tackled only theoretical problems. On the other hand, too many of those involved in applications are unaware of the useful theorems or are victims of the myth of the elusive Liapunov function. Our aim, in writing this book, has been twofold: to describe the present state of the most useful parts of the theory, and to appeal to the practical man with a wealth of applications taken from many varied fields.

Chapters I and II constitute an elementary self-contained treatment of stability theory. They should normally be read first. Almost every other chapter can be studied without further prerequisite, except that some definitions or propositions of Chapter VI are needed in Chapters VII, VIII, and IX. The whole of Chapter VI is used in Section IX.6.

We are also grateful to M. Everard, S. Spinacci and Kate MacDougall for their particularly expert typing of successive versions of the manuscript. Finally, it is a pleasure to acknowledge the financial support of this work by "Fonds National de la Recherche Scientifique".

Louvain-la-Neuve, October, 1975

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SOME NOTATIONS AND DEFINITIONS

This books requires a familiarity with some basic concepts from the theory of ordinary differential equations. As a general rule we have used symbols which are common place in mathematics. Let us however point out the following notations:

$\mathscr{R}$, the set of real numbers,

$\overline{\mathscr{R}}$, the extended real number system,

$a \geq 0$, a is a positive real number,

$a > 0$, a is a strictly positive real number,

$[a,b]$, closed interval,

$]a,b[$, open interval,

$(a|b)$ or $a^T b$, according to context, scalar product in $\mathscr{R}^n$,

$||x||$, norm of point x in $\mathscr{R}^n$,

$d(x,M) = \inf_{y \in M} ||x-y||$, distance from $x \in \mathscr{R}^n$ to $M \subset \mathscr{R}^n$,

$B_\varepsilon = \{x \in \mathscr{R}^n, ||x|| < \varepsilon\}$, open ball with center at the origin and radius $\varepsilon > 0$,

$B(a,\varepsilon) = \{x \in \mathscr{R}^n, ||x-a|| < \varepsilon\}$, open ball with center $a \in \mathscr{R}^n$ and radius $\varepsilon > 0$,

$B(M,\varepsilon) = \{x \in \mathscr{R}^n, d(x,M) < \varepsilon\}$, ε - neighborhood of the set $M \subset \mathscr{R}^n$,

$M_\varepsilon = B(M,\varepsilon) \cap \Omega$, ε - neighborhood of $M \in \mathscr{R}^n$ with respect to $\Omega \subset \mathscr{R}^n$,

E, unit $n \times n$ matrix,

$\dot{x} = \frac{dx}{dt}$, time derivative of the function $x: A \subset \mathscr{R} \to \mathscr{R}^n$,

$\frac{\partial f}{\partial x}$, jacobian matrix of the function $f: \mathscr{R}^n \to \mathscr{R}^m$, $x \to f(x)$,

J^+, see p. 7,

$\mathscr{K}$, see definition p. 12.

$\forall x$, universal quantifier; read "for all x" or "given x",

$\exists x$, existential quantifier; read "for some x" or "there exist x".

For general concepts on differential equations which are not defined in this text we refer to Ph. Hartman [1964], E. Coddington and N. Levinson [1965] or N. Rouche and J. Mawhin [1973]. The following definitions might be useful.

Let $A \subset \mathscr{R}$ and $f: A \to \mathscr{R}$, $x \to f(x)$ be a real valued function.

The function f is said to be:

increasing if $\forall x \in A$, $\forall y \in A$, $x < y$ implies $f(x) \leq f(y)$; i.e., for all x and y in A, $x < y$ implies $f(x) \leq f(y)$.

strictly increasing if $\forall x \in A$, $\forall y \in A$, $x < y$ implies $f(x) < f(y)$,

decreasing if $\forall x \in A$, $\forall y \in A$, $x < y$ implies $f(x) \geq f(y)$,

strictly decreasing if $\forall x \in A$, $\forall y \in A$, $x < y$ implies $f(x) > f(y)$,

monotonic if it is increasing on A or decreasing on A.

Let $a \in \bar{A}$, the extended closure of A. Then the limit superior (upper limit) of f at a is

$$\limsup_{x\to a} f(x) = \inf_{\delta>0}\{\sup\{f(x): x \in B(a,\delta),\ x \neq a\}\} \in \bar{\mathscr{R}}.$$

Similarly the limit inferior (lower limit) of f at a is

$$\liminf_{x\to a} f(x) = \sup_{\delta>0}\{\inf\{f(x): x \in B(a,\delta),\ x \neq a\}\} \in \bar{\mathscr{R}}.$$

If $a \in A$, the function f is said to be lower semi-continuous at a if $\liminf_{x\to \alpha} f(x) \geq f(a)$. If

$\limsup_{x \to a} f(x) \leq f(a)$, the function f is said to be <u>upper semi-continuous at</u> a. It is easy to verify that a function f is continuous at a if and only if it is lower and upper semi-continuous at a.

A function $V\colon \mathscr{R}^{1+n} \to \mathscr{R}$, $(t,x) \to V(t,x)$ is said to be <u>positive definite</u> (with respect to x) if there exists a function $a \in \mathscr{K}$ such that

(i) $V(t,0) = 0$

(ii) $V(t,x) \geq a(||x||)$.

If $-V$ is positive definite, the function V is said to be <u>negative definite</u> (with respect to x). If $V(t,0) = 0$ and $V(t,x) \geq 0$ the function V is said to be <u>positive semi-definite</u> (with respect to x). A function $V\colon \mathscr{R}^{1+n+m} \to \mathscr{R}$, $(t,x,y) \to V(t,x,y)$ is said to be <u>positive definite with respect to</u> x if for some function $a \in \mathscr{K}$

(i) $V(t,0,0) = 0$

(ii) $V(t,x,y) \geq a(||x||)$.

An important class of positive definite functions are the <u>positive quadratic forms</u>

$$V(x) = x^T A x$$

where A is a symmetric positive definite matrix (T denotes transpose).

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CHAPTER I

ELEMENTS OF STABILITY THEORY

The first two chapters are of an introductory character. Of the matters they exhibit, some have been known for a long time, others belong to the last fifteen years. Almost all will be considered over again in subsequent chapters, where the results will be extended or deepened. However, the next few pages are meant to give a fair idea of what stability and Liapunov's direct method are. Further, they should prove helpful to those concerned with simple practical applications. Of course, the rest of the book has been written to cope with less simple applications and, unfortunately or not, everyday practice proves how numerous they are ...

1. A First Glance at Stability Concepts

1.1. The English adjective "stable" originates from the Latin "stabilis", deriving itself from "stare", to stand. Its first acceptation is "standing firmly", "firmly established". A natural extension is "durable", not to mention the moral meaning "steady in purpose, constant". As it is, this concept of stability seems to be clear and of good use in everyday life. The layman might well wonder what reasons can be invoked to refine or complicate it. There are many, as we shall see.

Very early, the stability concept was specialized in mechanics to describe some type of equilibrium of a material particle or system. Consider for instance a particle subject to some forces and possessing an equilibrium point q_0. The equilibrium is called stable if, after any sufficiently small perturbations of its position and velocity, the particle remains forever arbitrarily near q_0, with arbitrarily small velocity. We shall not dwell on the well known example of a simple pendulum, whose lowest position, associated with zero velocity, is a stable equilibrium, whereas the highest one, also with zero velocity, is an unstable one.

Formulated in precise mathematical terms, this mechanical definition of stability was found useful in many situations, but inadequate in many others. This is why, with passing years, a host of other concepts have been introduced, each of them more or less related to the first definition and to the common sense meaning of stability. They were created either for definite technical or physical purposes, or for reasons of symmetry or completeness of the theory, or else to suit the fancy of their inventors. Later in this book (Chapter VI), we shall try, with much care, to separate the wheat from the chaff.

1.2. As contrasted with mechanical stability, the other concept known as Liapunov's stability has the following features: first, it pertains no more to a material particle (or the equations thereof), but to a general differential equation; second, it applies to a solution, i.e. not only to an equilibrium or critical point.

Let

$$\dot{x} = f(t,x), \tag{1.1}$$

where x and f are real n-vectors, t is the time (a real variable), f is defined on $\mathscr{R} \times \mathscr{R}^n$ and $\dot{x} = dx/dt$. We assume f smooth enough to ensure existence, uniqueness and continuous dependence of the solutions of the initial value problem associated with (1.1) over $\mathscr{R} \times \mathscr{R}^n$. For simplicity, we assume further that all solutions to be mentioned below exist for every $t \in \mathscr{R}$. Let $||\cdot||$ designate any norm on $\mathscr{R}^n$.

A solution $\bar{x}(t)$ of (1.1) is called <u>stable at</u> t_0, or, more precisely, <u>stable at</u> $t = t_0$ in the sense of A.M. Liapunov [1892] if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that if $x(t)$ is any other solution with $||x(t_0) - \bar{x}(t_0)|| < \delta$, then $||x(t) - \bar{x}(t)|| < \varepsilon$ for all $t \geq t_0$. Otherwise, of course, $\bar{x}(t)$ is called <u>unstable at</u> t_0.

Thus, it turns out that stability at t_0 is nothing but continuous dependence of the solutions on $x_0 = x(t_0)$, uniform with respect to $t \in [t_0,\infty[$.

1.3. <u>Exercise</u>. Prove that stability at t_0 implies stability at any other initial time (usually with different values for δ).

Hint: use the fact that, if $x(t;t_0,x_0)$ is the solution passing through x_0 at t_0, then the mapping

$$x(t;t_0,\cdot): x_0 \to x(t;t_0,x_0)$$

is a homeomorphism; i.e., it and its inverse are one to one and continuous.

1.4. We may gain some geometrical insight into this stability concept by considering again a pendulum, whose equation is $\ddot{x} + \omega^2 \sin x = 0$, with x and $\omega \in \mathscr{R}$. This second order equation is equivalent to the first order system

$$\dot{x} = y$$
$$\dot{y} = -\omega^2 \sin x.$$

As is well known, the origin of the (x,y)-plane is a center, i.e. all the solutions starting near the origin form a family of non-intersecting closed orbits encircling the origin. Given $\varepsilon > 0$, consider an orbit entirely contained in the disk B_ε of radius ε with center at the origin. Further, choose any other disk B_δ of radius δ, contained in this orbit. Clearly, every solution starting in B_δ at any initial time remains in B_ε. This demonstrates stability of the origin for any initial time.

On the other hand however, any other solution corresponding to one of the closed orbits is unstable. In fact, the period of the solution varies with the orbit and two points of the (x,y)-plane, very close to each other at $t = t_0$, but belonging to different orbits, will appear in opposition after some time. This happens however small the difference between periods. But it remains that, in some sense, the orbits are closed to each other. Similar examples led to a new concept called orbital stability, to be discussed later in this book, in connection with the stability of sets of points.

1.5. To say a little more about possible variations on the theme of stability, notice that in the case of the pendulum, the equilibrium $x = y = 0$ is such that no neighbouring solution approaches it when $t \to \infty$, as it would do if some appropriate friction were present. In many practical situations, it is useful to require, besides mere Liapunov stability of a solution $\bar{x}(t)$, that all neighbouring solutions $x(t)$ tend to $\bar{x}(t)$ when $t \to \infty$. This leads to the notion of asymptotic stability.

1.6. Many other examples can illustrate the necessity of creating new specific concepts. The last one to be mentioned here will be borrowed from celestial mechanics. Following common sense, the solar system is called stable if it is "durable" (cf. 1.1), i.e. if none of its constituent bodies escapes to infinity, and further if no two such bodies meet each other. But the velocities are unbounded if and only if two bodies approach each other. Therefore, stability in this sense (it is called Lagrange stability), simply means that the coordinates and velocities of the bodies are bounded. Boundedness of solutions thus appears as a legitimate and natural type of stability.

In the next section, we introduce a small number of definitions, in fact the most widely used and studied.

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