CLASSICAL MECHANICS

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PREFACE

An advanced course in classical mechanics has long been a time-honored part of the graduate physics curriculum. The present-day function of such a course, however, might well be questioned. It introduces no new physical concepts to the graduate student. It does not lead him directly into current physics research. Nor does it aid him, to any appreciable extent, in solving the practical mechanics problems he encounters in the laboratory.

Despite this arraignment, classical mechanics remains an indispensable part of the physicist's education. It has a twofold role in preparing the student for the study of modern physics. First, classical mechanics, in one or another of its advanced formulations, serves as the springboard for the various branches of modern physics. Thus, the technique of action-angle variables is needed for the older quantum mechanics, the Hamilton-Jacobi equation and the principle of least action provide the transition to wave mechanics, while Poisson brackets and canonical transformations are invaluable in formulating the newer quantum mechanics. Secondly, classical mechanics affords the student an opportunity to master many of the mathematical techniques necessary for quantum mechanics while still working in terms of the familiar concepts of classical physics.

Of course, with these objectives in mind, the traditional treatment of the subject, which was in large measure fixed some fifty years ago, is no longer adequate. The present book is an attempt at an exposition of classical mechanics which does fulfill the new requirements. Those formulations which are of importance for modern physics have received emphasis, and mathematical techniques usually associated with quantum mechanics have been introduced wherever they result in increased elegance and compactness. For example, the discussion of central force motion has been broadened to include the kinematics of scattering and the classical solution of scattering problems. Considerable space has been devoted to canonical transformations, Poisson bracket formulations, Hamilton-Jacobi theory, and action-angle variables. An introduction has been provided to the variational principle formulation of continuous systems and fields. As an illustration of the application of new mathematical techniques, rigid body rotations are treated from the standpoint of matrix transformations. The familiar Euler's theorem on the motion of a rigid body can then be presented in terms of the eigenvalue problem for an orthogonal matrix. As a consequence, such diverse topics as the inertia tensor, Lorentz transformations in Minkowski space, and resonant frequencies of small oscillations become capable of a unified mathematical treatment. Also, by this technique it becomes possible to include at an early stage the difficult concepts of reflection operations and pseudotensor quantities, so important in modern quantum mechanics. A further advantage of matrix methods is that "spinors" can be introduced in connection with the properties of Cayley-Klein parameters.

Several additional departures have been unhesitatingly made. All too often, special relativity receives no connected development except as part of a highly specialized course which also covers general relativity. However, its vital importance in modern physics requires that the student be exposed to special relativity at an early stage in his education. Accordingly, Chapter 6 has been devoted to the subject. Another innovation has been the inclusion of velocity-dependent forces. Historically, classical mechanics developed with the emphasis on static forces dependent on position only, such as gravitational forces. On the other hand, the velocity-dependent electromagnetic force is constantly encountered in modern physics. To enable the student to handle such forces as early as possible, velocity-dependent potentials have been included in the structure of mechanics from the outset, and have been consistently developed throughout the text.

Still another new element has been the treatment of the mechanics of continuous systems and fields in Chapter 11, and some comment on the choice of material is in order. Strictly interpreted, the subject could include all of elasticity, hydrodynamics, and acoustics, but these topics lie outside the prescribed scope of the book, and adequate treatises have been written for most of them. In contrast, no connected account is available on the classical foundations of the variational principle formulation of continuous systems, despite its growing importance in the field theory of elementary particles. The theory of fields can be carried to considerable length and complexity before it is necessary to introduce quantization. For example, it is perfectly feasible to discuss the stress-energy tensor, microscopic equations of continuity, momentum space representations, etc., entirely within the domain of classical physics. It was felt, however, that an adequate discussion of these subjects would require a sophistication beyond what could naturally be expected of the student. Hence it was decided, for this edition at least, to limit Chapter 11 to an elementary description of the Lagrangian and Hamiltonian formulation of fields.

The course for which this text is designed normally carries with it a prerequisite of an intermediate course in mechanics. For both the inadequately prepared graduate student (an all too frequent occurrence) and the ambitious senior who desires to omit the intermediate step, an effort was made to keep the book self-contained. Much of Chapters 1 and 3 is therefore devoted to material usually covered in the preliminary courses.

With few exceptions, no more mathematical background is required of the student than the customary undergraduate courses in advanced calculus and vector analysis. Hence considerable space is given to developing the more complicated mathematical tools as they are needed. An elementary acquaintance with Maxwell's equations and their simpler consequences is necessary for understanding the sections on electromagnetic forces. Most entering graduate students have had at least one term's exposure to modern physics, and frequent advantage has been taken of this circumstance to indicate briefly the relation between a classical development and its quantum continuation.

A large store of exercises is available in the literature on mechanics, easily accessible to all; and there consequently seemed little point to reproducing an extensive collection of such problems. The exercises appended to each chapter therefore have been limited, in the main, to those which serve as extensions of the text, illustrating some particular point or proving variant theorems. Pedantic museum pieces have been studiously avoided.

The question of notation is always a vexing one. It is impossible to achieve a completely consistent and unambiguous system of notation that is not at the same time impracticable and cumbersome. The customary convention has been followed of indicating vectors by bold face Roman letters. In addition, matrix quantities of whatever rank, and tensors other than vectors, are designated by bold face sans serif characters, thus: **A**. An index of symbols is appended at the end of the book, listing the initial appearance of each meaning of the important symbols. Minor characters, appearing only once, are not included.

References have been listed at the end of each chapter, for elaboration of the material discussed or for treatment of points not touched on. The evaluations accompanying these references are purely personal, of course, but it was felt necessary to provide the student with some guide to the bewildering maze of literature on mechanics. These references, along with many more, are also listed at the end of the book. The list is not intended to be in any way complete, many of the older books being deliberately omitted. By and large, the list contains the references used in writing this book, and must therefore serve also as an acknowledgement of my debt to these sources.

The present text has evolved from a course of lectures on classical mechanics that I gave at Harvard University, and I am grateful to Professor

J. H. Van Vleck, then Chairman of the Physics Department, for many personal and official encouragements. To Professor J. Schwinger, and other colleagues I am indebted for many valuable suggestions. I also wish to record my deep gratitude to the students in my courses, whose favorable reaction and active interest provided the continuing impetus for this work. תושלב״ע

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CONTENTS

CHAPTER 1

SURVEY OF THE ELEMENTARY PRINCIPLES

The motion of material bodies formed the subject of some of the earliest researches pursued by the pioneers of physics. From their efforts there has evolved a vast field known as analytical mechanics or dynamics, or simply, mechanics. In the present century the term "classical mechanics" has come into wide use to denote this branch of physics in contradistinction to the newer physical theories, especially quantum mechanics. We shall follow this usage, interpreting the name to include the type of mechanics arising out of the special theory of relativity. It is the purpose of this book to develop the structure of classical mechanics and to outline some of its applications of present-day interest in pure physics.

Basic to any presentation of mechanics are a number of fundamental physical concepts, such as space, time, simultaneity, mass, and force. In discussing the special theory of relativity the notions of simultaneity and of time and length scales will be examined briefly. For the most part, however, these concepts will not be analyzed critically here; rather, they will be assumed as undefined terms whose meanings are familiar to the reader.

1–1 Mechanics of a particle. The essential physics involved in the mechanics of a particle is contained in *Newton's Second Law of Motion*, which may be considered equivalently as a fundamental postulate or as a definition of force and mass. For a single particle, the correct form of the law is:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \tag{1-1}$$

where $\mathbf{F}$ is the total force acting on the particle and $\mathbf{p}$ is the *linear momentum* of the particle defined as follows: Let s be the curve traced by the particle in its motion, and $\mathbf{r}$ the radius vector from the origin to the particle. The vector velocity can then be defined formally by the equation:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}, \tag{1-2}$$

where the derivative is evaluated by the usual limiting process (cf. Fig. 1–1):

$$\frac{d\mathbf{r}}{dt} = \underset{\Delta t \to 0}{\mathrm{L}} \frac{\mathbf{r}_2 - \mathbf{r}_1}{\Delta t} = \underset{\Delta t \to 0}{\mathrm{L}} \frac{\Delta \mathbf{s}}{\Delta t} = \frac{d\mathbf{s}}{dt}.$$

(This last form for the derivative explicitly indicates that **v** is tangent to the curve.) Then the linear momentum **p** is defined in terms of the velocity as

$$\mathbf{p} = m\mathbf{v}, \tag{1-3}$$

so that (1-1) can be written

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}). \tag{1-4}$$

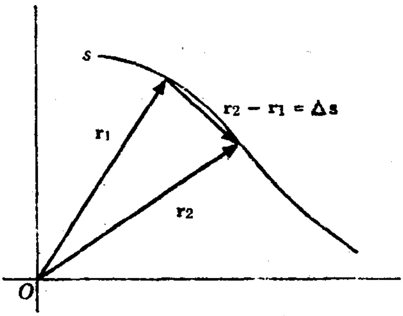

FIG. 1-1. Motion of a particle in space, illustrating the definition of velocity.

In most cases the mass of the particle is constant and Eq. (1-1) reduces to:

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}, \tag{1-5}$$

where **a** is called the acceleration of the particle and is defined by

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}. \tag{1-6}$$

Many of the important conclusions of mechanics can be expressed in the form of conservation theorems, which indicate under what conditions various mechanical quantities are constant in time. Eq. (1-1) directly furnishes the first of these, the

Conservation Theorem for the Linear Momentum of a Particle: If the total force, **F**, *is zero then* $\dot{\mathbf{p}} = 0$ *and the linear momentum,* **p**, *is conserved.*

The angular momentum of the particle about point O, denoted by **L**, is defined as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \tag{1-7}$$

where **r** is the radius vector from O to the particle. Notice that the order of the factors is important. We now define the *moment of force* or *torque* about O as

$$\mathbf{N} = \mathbf{r} \times \mathbf{F}. \tag{1-8}$$

The equation analogous to (1-1) for **N** is obtained by forming the cross product of **r** with Eq. (1-4):

$$\mathbf{r} \times \mathbf{F} = \mathbf{N} = \mathbf{r} \times \frac{d}{dt}(m\mathbf{v}). \tag{1-9}$$

Eq. (1-9) can be written in a different form by using the vector identity:

$$\frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times \frac{d}{dt}(m\mathbf{v}),$$

where the first term on the right obviously vanishes. In consequence of this identity Eq. (1-9) takes the form

$$\mathbf{N} = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = \frac{d\mathbf{L}}{dt}. \tag{1-10}$$

Note that both **N** and **L** depend upon the point O, about which the moments are taken.

As was the case for Eq. (1–1), the torque equation, (1–10), also yields an immediate conservation theorem, this time the

Conservation Theorem for the Angular Momentum of a Particle: If the total torque, **N**, *is zero then* $\dot{\mathbf{L}} = 0$, *and the angular momentum* **L** *is conserved.*

Next consider the work done by the external force **F** upon the particle in going from point 1 to point 2. By definition this work is

$$W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{s}. \tag{1–11}$$

For constant mass (as will be assumed from now on unless otherwise specified), the integral in Eq. (1–11) reduces to

$$\int \mathbf{F} \cdot d\mathbf{s} = m \int \frac{d\mathbf{v}}{dt} \cdot \mathbf{v}\, dt = \frac{m}{2} \int \frac{d}{dt}(v^2)\, dt,$$

and therefore

$$W_{12} = \frac{m}{2}(v_2^2 - v_1^2). \tag{1–12}$$

The scalar quantity $mv^2/2$ is called the kinetic energy of the particle and is denoted by T, so that the work done is equal to the change in the kinetic energy:

$$W_{12} = T_2 - T_1. \tag{1–13}$$

If the force field is such that the work W done around a closed orbit is zero, i.e.,

$$\oint \mathbf{F} \cdot d\mathbf{s} = 0, \tag{1–14}$$

then the force (and the system) is said to be *conservative*. Physically it is clear that a system cannot be conservative if friction or other dissipation forces are present, for $\mathbf{F} \cdot d\mathbf{s}$ due to friction is always positive and the integral cannot vanish. By Stokes' Theorem, the condition for conservative forces, Eq. (1–14), can be written:

$$\nabla \times \mathbf{F} = 0,$$

and since the curl of a gradient always vanishes **F** must therefore be the gradient of some scalar:

$$\mathbf{F} = -\nabla V, \tag{1–15}$$

where V is called the *potential*, or *potential energy*. The existence of V can be established without the use of theorems of vector calculus. If Eq. (1–14) holds, the work W_{12} must be independent of the path of integration between end points 1 and 2. It follows then that it must be possible to express W_{12} as the change in a quantity which depends only upon the positions of the end points. This quantity may be designated by $-V$, so that for a differential path length we have the relation:

$$\mathbf{F} \cdot d\mathbf{s} = -dV$$

or

$$F_s = -\frac{\partial V}{\partial s},$$

which is equivalent to Eq. (1–15). Note that in Eq. (1–15) we can add to V any quantity constant in space, without affecting the results. Hence, *the zero level of V is arbitrary.*

For a conservative system the work done by the forces is

$$W_{12} = V_1 - V_2. \tag{1–16}$$

Combining Eq. (1–16) with Eq. (1–13) we have the result

$$T_1 + V_1 = T_2 + V_2, \tag{1–17}$$

which states in symbols the

Energy Conservation Theorem for a Particle: If the forces acting on a particle are conservative, then the total energy of the particle, $T + V$, is conserved.

1–2 Mechanics of a system of particles. In generalizing the ideas of the previous section to systems of many particles, we must distinguish between the *external forces* acting on the particles due to sources outside the system, and *internal forces* on, say, some particle i due to all other particles in the system. Thus the equation of motion (Newton's Second Law) for the ith particle is to be written:

$$\sum_j \mathbf{F}_{ji} + \mathbf{F}_i^{(e)} = \dot{\mathbf{p}}_i, \tag{1–18}$$

where $\mathbf{F}_i^{(e)}$ stands for an external force, and $\mathbf{F}_{ji}$ is the internal force on the ith particle due to the jth particle ($\mathbf{F}_{ii}$, naturally, is zero). We shall assume that the $\mathbf{F}_{ij}$ (like the $\mathbf{F}_i^{(e)}$) obey Newton's third law of action and reaction: that the forces two particles exert on each other are equal and opposite, *and lie along the line joining them.* There are some important systems in which the forces do not follow this law, notably the electromagnetic forces between moving particles. The theorems derived below must be applied to such systems with due caution.

Summed over all particles Eq. (1–18) takes the form:

$$\frac{d^2}{dt^2} \sum_i m_i \mathbf{r}_i = \sum_i \mathbf{F}_i^{(e)} + \sum_{\substack{i,j \\ i \neq j}} \mathbf{F}_{ji}. \tag{1–19}$$

The first sum on the right is simply the total external force $\mathbf{F}^{(e)}$, while the second term vanishes, since the law of action and reaction states that each pair $\mathbf{F}_{ij} + \mathbf{F}_{ji}$ is zero. To reduce the left-hand side we define a vector $\mathbf{R}$ as the average of the radii vectors of the particles, weighted in proportion to their mass, i.e.,*

$$\mathbf{R} = \frac{\Sigma m_i \mathbf{r}_i}{\Sigma m_i} = \frac{\Sigma m_i \mathbf{r}_i}{M}. \tag{1-20}$$

The vector $\mathbf{R}$ defines a point known as the *center of mass*, or more loosely as the center of gravity, of the system. With this definition (1–19) reduces to

$$M \frac{d^2\mathbf{R}}{dt^2} = \sum_i \mathbf{F}_i^{(e)} \equiv \mathbf{F}^{(e)}, \tag{1-21}$$

which states that the center of mass moves as if the total external force were acting on the entire mass of the system concentrated at the center of mass. Purely internal forces therefore have no effect on the motion of the center of mass. An oft-quoted example is the motion of an exploding shell; the center of mass of the fragments traveling as if the shell were still in a single piece (neglecting air resistance). The same principle is involved in jet and rocket propulsion. In order that the motion of the center of mass be unaffected, the ejection of the exhaust gases at high velocity must be counterbalanced by the forward motion of the vehicle.

By Eq. (1–20) the total linear momentum of the system,

$$\mathbf{P} = \Sigma m_i \frac{d\mathbf{r}_i}{dt},$$

is the total mass of the system times the velocity of the center of mass. Consequently, the equation of motion for the center of mass, (1–21), can be restated as the

Conservation Theorem for the Linear Momentum of a System of Particles: If the total external force is zero, the total linear momentum is conserved.

We obtain the total angular momentum of the system by forming the cross product $\mathbf{r}_i \times \mathbf{p}_i$ and summing over i. If this operation is performed in Eq. (1–18) there results

$$\sum_i (\mathbf{r}_i \times \dot{\mathbf{p}}_i) = \sum_i \frac{d}{dt}(\mathbf{r}_i \times \mathbf{p}_i) = \dot{\mathbf{L}} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(e)} + \sum_{\substack{i,j\\ i \neq j}} \mathbf{r}_i \times \mathbf{F}_{ji}. \tag{1-22}$$

* This definition may be more familiar if Eq. (1–20) is written in terms of the cartesian coordinates

$$X = \frac{\Sigma m_i x_i}{\Sigma m_i}, \qquad Y = \frac{\Sigma m_i y_i}{\Sigma m_i}, \qquad Z = \frac{\Sigma m_i z_i}{\Sigma m_i}.$$

The last term on the right in (1–22) can be considered a sum of the pairs of the form

$$\mathbf{r}_i \times \mathbf{F}_{ji} + \mathbf{r}_j \times \mathbf{F}_{ij} = (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ji}, \tag{1–23}$$

using the equality of action and reaction. But $\mathbf{r}_i - \mathbf{r}_j$ is identical with the vector $\mathbf{r}_{ij}$ from j to i, and the law of action and reaction further states that

$$\mathbf{r}_{ij} \times \mathbf{F}_{ji} = 0,$$

since $\mathbf{F}_{ji}$ is along the line between the two particles. Hence this sum vanishes and (1–22) may be written

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}^{(e)}. \tag{1–24}$$

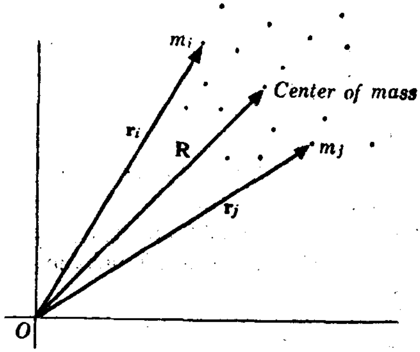

FIG. 1–2. The center of mass of a system of particles.

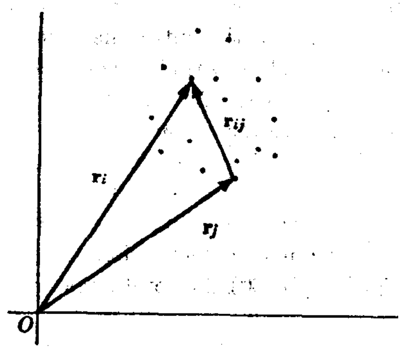

FIG. 1–3. The vector $\mathbf{r}_{ij}$ between the ith and jth particles.

The time derivative of the total angular momentum is thus equal to the moment of the external force about the given point. Corresponding to Eq. (1–24) is the

Conservation Theorem for Total Angular Momentum: **L** *is constant in time if the applied (external) torque is zero.*

(It is perhaps worthwhile to emphasize that this is a *vector* theorem, i.e., L_z will be conserved if $N_z^{(e)}$ is zero, even if $N_x^{(e)}$ and $N_y^{(e)}$ are not zero.)

Note that the conservation of angular momentum of a system in the absence of applied torques holds only if the law of action and reaction is valid. In a system involving moving charges, where this law is violated, it is not the total mechanical angular momentum which is conserved, but rather the sum of the mechanical and the electromagnetic "angular momentum" of the field.

Eq. (1-21) states that the total linear momentum of the system is the same as if the entire mass were concentrated at the center of mass and moving with it. The analogous theorem for angular momentum is more complicated. With the origin O as reference point the total angular momentum of the system is

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i.$$

Let $\mathbf{R}$ be the radius vector from O to the center of mass, and let $\mathbf{r}'_i$ be the radius vector from the center of mass to the ith particle. Then we have (cf. Fig. 1-4):

$$\mathbf{r}_i = \mathbf{r}'_i + \mathbf{R} \qquad (1\text{-}25)$$

and

$$\mathbf{v}_i = \mathbf{v}'_i + \mathbf{v},$$

where

$$\mathbf{v} = \frac{d\mathbf{R}}{dt},$$

is the velocity of the center of mass relative to O, and

$$\mathbf{v}' = \frac{d\mathbf{r}'_i}{dt},$$

is the velocity of the ith particle relative to the center of mass of the system.

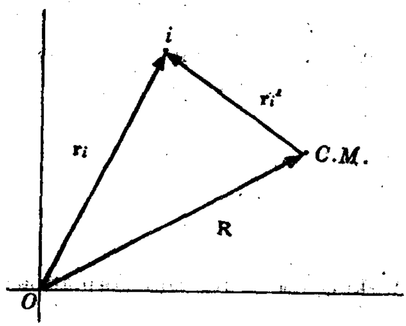

Fig. 1-4. The vectors involved in the shift of reference point for the angular momentum.

Using Eq. (1-25), the total angular momentum takes on the form

$$\mathbf{L} = \sum_i \mathbf{R} \times m_i\mathbf{v} + \sum_i \mathbf{r}'_i \times m_i\mathbf{v}'_i + \left(\sum_i m_i\mathbf{r}'_i\right) \times \mathbf{v} + \mathbf{R} \times \frac{d}{dt} \sum_i m_i\mathbf{r}'_i.$$

The last two terms in this expression vanish, for both contain the factor $\Sigma m_i\mathbf{r}'_i$, which, it will be recognized, defines the radius vector of the center of mass in the very coordinate system whose origin is the center of mass, and is therefore a null vector. Rewriting the remaining terms, the total angular momentum about O is:

$$\mathbf{L} = \mathbf{R} \times M\mathbf{v} + \sum_i \mathbf{r}'_i \times \mathbf{p}'_i. \qquad (1\text{-}26)$$

In words, Eq. (1-26) says that the total angular momentum about a point O is the angular momentum of the system concentrated at the center of mass, plus the angular momentum of motion about the center of mass. The form of Eq. (1-26) emphasizes that in general $\mathbf{L}$ depends on the origin O, through the vector $\mathbf{R}$. Only if the center of mass is at rest with respect to O will the angular momentum be independent of the point of reference. In this case the first term in (1-26) vanishes, and $\mathbf{L}$ always reduces to the angular momentum taken about the center of mass.

Finally, let us consider the energy equation. As in the case of a single particle, we calculate the work done by all forces in moving the system from an initial configuration 1, to a final configuration 2:

$$W_{12} = \sum_i \int_1^2 \mathbf{F}_i \cdot d\mathbf{s}_i = \sum_i \int_1^2 \mathbf{F}_i^{(e)} \cdot d\mathbf{s}_i + \sum_{\substack{i,j \\ i \neq j}} \int_1^2 \mathbf{F}_{ji} \cdot d\mathbf{s}_i. \tag{1-27}$$

Again, the equations of motion can be used to reduce the integrals to

$$\sum_i \int_1^2 \mathbf{F}_i \cdot d\mathbf{s}_i = \sum_i \int_1^2 m_i \dot{\mathbf{v}}_i \cdot \mathbf{v}_i \, dt = \sum_i \int_1^2 d(\tfrac{1}{2} m_i v_i^2).$$

Hence the work done can still be written as the difference of the final and initial kinetic energies:

$$W_{12} = T_2 - T_1,$$

where T, the total kinetic energy of the system, is

$$T = \frac{1}{2} \sum_i m_i v_i^2. \tag{1-28}$$

Making use of the transformations to center of mass coordinates, given in Eq. (1–25), we may write T also as

$$\begin{aligned} T &= \frac{1}{2} \sum_i m_i (\mathbf{v} + \mathbf{v}_i') \cdot (\mathbf{v} + \mathbf{v}_i') \\ &= \frac{1}{2} \sum_i m_i v^2 + \frac{1}{2} \sum_i m_i v_i'^2 + \mathbf{v} \cdot \frac{d}{dt} \left(\sum_i m_i \mathbf{r}_i' \right), \end{aligned}$$

and by the reasoning already employed in calculating the angular momentum, the last term vanishes, leaving

$$T = \frac{1}{2} M v^2 + \frac{1}{2} \sum_i m_i v_i'^2. \tag{1-29}$$

The kinetic energy, like the angular momentum, thus also consists of two parts: the kinetic energy obtained if all the mass were concentrated at the center of mass, plus the kinetic energy of motion about the center of mass.

Consider now the right-hand side of Eq. (1–27). In the special case that the external forces are derivable from a potential the first term can be written as

$$\sum_i \int_1^2 \mathbf{F}_i^{(e)} \cdot d\mathbf{s}_i = - \sum_i \int_1^2 \nabla_i V_i \cdot d\mathbf{s}_i = - \sum_i V_i \Big|_1^2$$

where the subscript i on the del operator indicates that the derivatives are with respect to the components of $\mathbf{r}_i$. If the internal forces are also con-