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Mathematical Theory of Incompressible Nonviscous Fluids

不可压缩非粘性流的数学理论

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Preface

Fluid dynamics is an ancient science incredibly alive today. Modern technology and new needs require a deeper knowledge of the behavior of real fluids. and new discoveries or steps forward pose. quite often, challenging and difficult new mathematical problems. In this framework, a special role is played by incompressible nonviscous (sometimes called *perfect*) flows. This is a mathematical model consisting essentially of an evolution equation (the Euler equation) for the velocity field of fluids. Such an equation, which is nothing other than the Newton laws plus some additional structural hypotheses, was discovered by Euler in 1755, and although it is more than two centuries old, many fundamental questions concerning its solutions are still open. In particular, it is not known whether the solutions, for reasonably general initial conditions, develop singularities in a finite time, and very little is known about the long-term behavior of smooth solutions. These and other basic problems are still open, and this is one of the reasons why the mathematical theory of perfect flows is far from being completed.

Incompressible flows have been attached, by many distinguished mathematicians, with a large variety of mathematical techniques so that, today, this field constitutes a very rich and stimulating part of applied mathematics. The idea of writing the present book was motivated by the fact that, although there are many interesting books on the subject, no recent one, to our knowledge, is oriented toward mathematical physics. By this we mean a book that is mathematically rigorous and as complete as possible without hiding the underlying physical ideas, presenting the arguments in a natural order, from basic questions to more sophisticated ones, proving everything and trying, at the same time, to avoid boring technicalities. This is our purpose.

The book does not require a deep mathematical knowledge. The required background is a good understanding of the classical arguments of mathematical analysis, including the basic elements of ordinary and partial differential equations, measure theory and analytic functions, and a few notions of potential theory and functional analysis.

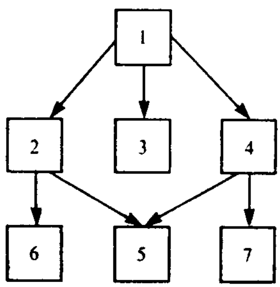
The exposition is as self-contained as possible. Several appendices, devoted to technical or elementary classical arguments, are included. This does not mean, however, that the book is easy to read. In fact, even if we tried to present the topics in an elementary fashion and in the simplest cases, the style is, in general, purely mathematical and rather concise, so that the reader quite often is requested to spend some time in independent thinking during the most delicate steps of the exposition. Some exercises, with a varying degree of difficulty (the most difficult are marked by *), are presented at the end of many chapters. We believe solving them is the best test to see whether the basic notions have been understood.

The choice of arguments is classical and in a sense obligatory. The presentation of the material, the relative weight of the various arguments, and the general style reflect the tastes of the authors and their knowledge. It cannot be otherwise.

The material is organized as follows: In Chapter 1 we present the basic equations of motion of incompressible nonviscous fluids (the Euler equation) and their elementary properties. In Chapter 2 we discuss the construction of the solutions of the Cauchy problem for the Euler equation. In Chapter 3 we study the stability properties of stationary solutions. In Chapter 4 we introduce and discuss the vortex model. In Chapter 5 we briefly analyze the approximation schemes for the solutions of fluid dynamical equations. Chapter 6 is devoted to the time evolution of discontinuities such as the vortex sheets or the water waves. Finally, in Chapter 7 we discuss turbulent motions. This last chapter mostly contains arguments of current research and is essentially discursive.

The final section of each chapter is generally devoted to a discussion of the existing literature and further developments. We hope that this will stimulate the reader to study and research further.

The book can be read following the natural order of the chapters, but also along the following paths:



A possible criticism of the book is that two-dimensional flows are treated in much more detail than three-dimensional ones, which are, physically speaking, much more interesting. Unfortunately, for a mathematical treatise, it cannot be otherwise: The mathematical theory of a genuine three-dimensional flow is, at present, still poor compared with the rather rich analysis of the two-dimensional case to which we address many efforts.

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Contents

CHAPTER 1

General Considerations on the Euler Equation

This chapter has an introductory nature, wherein we discuss the fundamental equations describing the motion of an incompressible nonviscous fluid and establish some elementary properties.

1.1. The Equation of Motion of an Ideal Incompressible Fluid

In this section we establish the mathematical model of an ideal incompressible fluid, deriving heuristically the equation governing its motion.

Fluid mechanics studies the behavior of gases and liquids. The phenomena we want to study are macroscopic: we do not want to investigate the dynamics of the individual molecules constituting the fluid, but the gross behavior of many of them. For this purpose we assume the fluid as a continuum, a point of which is a very small portion of the real fluid, negligible with respect to the macroscopic size (for instance, the size of the vessel containing the fluid), but very large with respect to the molecular length. This small volume, a point in our mathematical description, will be called *fluid particle* or *element of fluid* later in this book. As a consequence, the physical state of a fluid will be described by properties of the fluid particles and not by the physical state of all the microscopic molecules. The macroscopic fields describing the state, as, for instance, the velocity field, $u = u(x)$, the density field $\rho = \rho(x)$, the temperature field, $T = T(x)$, etc., can be physically interpreted (and, in principle, calculated) by means of averages of suitable microscopic quantities. For example, the macroscopic velocity field in a point $u(x)$ means

$$u(x) = \frac{1}{N(x)} \sum_{i=1}^{N(x)} \mu_i, \qquad (1.1)$$

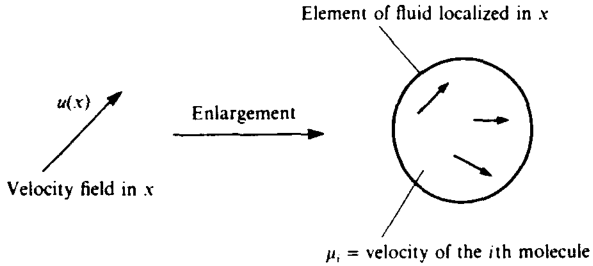

Figure 1.1

where $N(x)$ is the number of molecules associated to the fluid particle localized in x and μ_i, $i = 1, \ldots, N(x)$ are the velocities of these molecules (Fig. 1.1).

It would be very interesting to deduce the evolution equation for the fields, $u = u(x)$, $\rho = \rho(x)$, etc., starting from the Newton equation which governs the motion of the molecules. To give a measure of the difficulty of this program we note that the macroscopic observables u, ρ, T, etc., give us a reduced description of the physical system we are considering. Such a system is described, in much more detail, by the positions and the velocities of all the microscopic molecules. Therefore, it is not at all obvious that we are able to deduce some closed equations involving only the interesting observables.

Until now, a rigorous microscopic derivation of the fluid equations from the Newton laws is not known. For some discussion on this point we address the reader to Section 1.5, which is devoted to comments and bibliographical notes. In the absence of this deduction we limit ourselves to fixing the mathematical model of a fluid by heuristic considerations only, without taking into account its microscopic structure. We will deduce the basic equation, called the Euler equation, by the use of reasonable assumptions on the motion of the fluid parlicles. In the following sections, our study will be essentially deductive, starting from the Euler equation, which constitutes our mathematical model. Obviously, we will not neglect the physical interpretation which is important to verify the validity of the model itself and the relevance of the results.

The rest of the present section is devoted to the derivation of the Euler equation.

Let $D \subset \mathbb{R}^3$, an open and bounded set of the physical space with a regular boundary ∂D. D contains a fluid represented as a continuum of particles localized in any point $x \in D$.

An *incompressible displacement* of the fluid is a transformation $s: D \to D$ such that the following properties hold:

(a) s is invertible and $s(D) = D$;
(b) $s, s^{-1} \in C^1(D)$; and
(c) s preserves the Lebesgue measure.

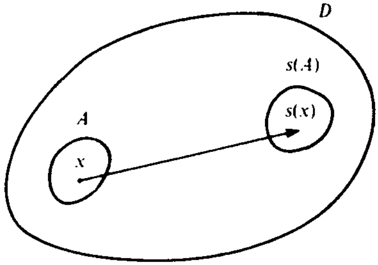

Figure 1.2

The property (c) means that, for any measurable set A, $A \subset D$, denoted by

$$s(A) = \{x \in D \mid s^{-1}(x) \in A\}, \tag{1.2}$$

we have

$$|s(A)| = |A|, \tag{1.3}$$

where $|A|$ = meas A denotes the Lebesgue measure of A (Fig. 1.2). We denote by S the set of all the incompressible displacements. It is evident that S has a group structure with respect to the law of natural composition

$$s_1 \circ s_2(x) = s_2(s_1(x)).$$

An incompressible motion is, by definition, a function s, $t \in R^1 \to \Phi_{s,t} \in S$ such that:

(1) $\Phi_{s,t}(\Phi_{t,r}(x)) = \Phi_{s,r}(x)$;
(2) $\Phi_{t,s}(\Phi_{s,t}(x)) = \Phi_{t,t}(x) = x$; and
(3) $\Phi_{t,s}(x)$ is continuously differentiable in t and s.

Here $\Phi_{t,s}$ denotes the position at time t of the particle of fluid that at time s was in x. We will denote by M, in the sequel, the family of incompressible motions.

We note that these conditions are reasonable properties of regularity. The requirement that the transformation be invertible means also that two different particles of fluid cannot occupy the same position. Moreover, the definition of Φ itself gives the conservation of the Lebesgue measure during the motion.

These conditions make it very easy to study the time evolution of the *density field* $\rho = \rho(x, t)$. We denote by $\rho(x, t)\,dx$ the mass of fluid contained in the element of volume dx at time t, and we assume that $\rho \in C^1(D)$. By the law of conservation of mass we have

$$\frac{d}{dt}\int_{V_t} \rho(x, t)\,dx = 0, \tag{1.4}$$

where

$$V_t = \{\Phi_t(x) | x \in V_0\} \tag{1.5}$$

is the region moving along the trajectories of an incompressible motion and $\Phi_t(x) = \Phi_{t,0}(x)$.

Let

$$u(\Phi_t(x), t) = \frac{d}{dt}\Phi_t(x) \tag{1.6}$$

be the velocity field associated with this motion.

By (1.4) we have

$$\begin{aligned} \frac{d}{dt}\int_{V_t} \rho(x, t)\, dx &= \frac{d}{dt}\int_{V_0} \rho(\Phi_t(x), t) J_t(x)\, dx \\ &= \frac{d}{dt}\int_{V_0} \rho(\Phi_t(x), t)\, dx = 0, \end{aligned} \tag{1.7}$$

where $J_t(x)$ is the Jacobian of the transformation $x \to \Phi_t(x)$. The incompressibility condition (together with the continuity of the transformation) implies that it is one.

Hence, by the arbitrariness of V_0, we have

$$\frac{d}{dt}\rho(\Phi_t(x), t) = (\partial_t + u \cdot \nabla)\rho(\Phi_t(x), t) = 0. \tag{1.8}$$

From a physical point of view there are interesting situations in which the density is initially (and hence by (1.8) for all times) not constant in space. We will provide an example in Chapter 6. However, in most of the physically relevant cases, in which the model of incompressible fluid applies, the density can be assumed to be essentially constant. In the present book we will assume the density to be always constant (for simplicity $\rho = 1$), unless explicitly mentioned otherwise.

The condition of incompressibility is equivalent, by a well-known theorem on differential equations (the Liouville Theorem, see Appendix 1.1), to the condition

$$\operatorname{div} u(x, t) \equiv \nabla \cdot u(x, t) = 0, \qquad \forall x \in D, \quad t \in \mathbb{R}. \tag{1.9}$$

Equation (1.9) is usually called the continuity equation for incompressible flows.

From this point on, in this section, we are assuming $u \in C^1(D \times \mathbb{R}^1)$. Moreover, for any t, $u(x, t)$ is assumed continuous in $x \in \bar{D} \equiv D \cup \partial D$. This allows us to define the velocity $u(x, t)$ on the boundary ∂D as a limit.

We will now establish the boundary conditions. In general, for partial differential equations describing physical systems, the boundary conditions are a mathematical expression of the interaction of the system with the boundary. In our case, we must assume the most general and natural assumption which can be deduced from kinematic considerations only: the fluid particles cannot pass through the boundary so that

$$u(x,t)\cdot n = v(x)\cdot n \quad \text{on } \partial D, \tag{1.10}$$

where $v(x)$ is the velocity of the boundary at the point x. Most of the time, later in this book, we will comsider the container D at rest so that $v(x) = 0$ for all $x \in \partial D$.

Once the velocity field u is known, the trajectories $\Phi_t(x)$ can be uniquely built by solving the initial value problem (1.6) for the unknown quantity $\Phi_t(x)$ with initial value x at time $t = 0$.

We now want to state the equations of motion of an incompressible fluid. To determine the motion of the fluid particles we must specify the interactions among the particles themselves. We consider the only interaction produced by the incompressibility. This means that each particle tries to move freely, the only constraint being that it cannot occupy the site in which there is another particle. Later on we will be more precise. This model of an incompressible fluid is called *ideal* (or *perfect*) and it is the simplest model we can conceive.

To find the equations of motion it is convenient to consider the *Principle of Stationary Action* as suggested by the classical mechanics of systems with a finite number of the degrees of freedom.

The kinetic energy (and also the Lagrangian) of the system is given by the following expression:

$$E = \frac{1}{2}\int_D dx \left[\frac{d}{dt}\Phi_t(x)\right]^2. \tag{1.11}$$

So the action is defined as

$$A(\Phi; t_1, t_2) = \frac{1}{2}\int_{t_1}^{t_2} dt \int_D dx \left[\frac{d}{dt}\Phi_t(x)\right]^2. \tag{1.12}$$

Then $\Phi \to A(\Phi; t_1, t_2)$ is a functional defined on M, the space of incompressible motions. We have not added an interaction energy since the motion we have in mind is the same as the free motion, on a given manifold, of a finite particle system. In our case the "manifold" is given by the incompressibility constraint. Therefore, as in the mechanical analogue where the variation is chosen in accord with the constraint, here we will consider variations in the class M. Hence, to determine the physical motion Φ, we ask that the action be stationary for variations, $\Phi \to \Phi + \delta\Phi$, which are compatible with the constraint of incompressibility, and to satisfy $\delta\Phi_{t_1}(x) = \delta\Phi_{t_2}(x) = 0$ for all $x \in D$ (Fig. 1.3). Moreover, the variation must also satisfy the boundary conditions

$$\frac{d}{dt}\Phi_t^\varepsilon(x)\cdot n = 0, \qquad x \in \partial D.$$

We denote by Φ^ε, $\varepsilon \in [0, \varepsilon_0]$, a family of varied motions, tangent to the boundary ∂D, such that

$$\Phi^0 = \Phi, \qquad \Phi_{t_1}^\varepsilon = \Phi_{t_1}, \qquad \Phi_{t_2}^\varepsilon = \Phi_{t_2}, \qquad \forall \varepsilon \in [0, \varepsilon_0].$$

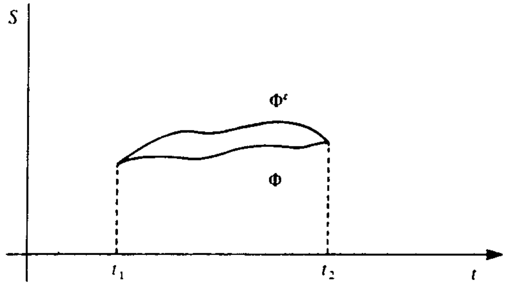

Figure 1.3

We impose that the action A be stationary on Φ, namely,

$$\frac{d}{d\varepsilon} A(\Phi^\varepsilon; t_1, t_2)|_{\varepsilon=0} = 0. \tag{1.13}$$

From (1.13) we easily obtain

$$\int_{t_1}^{t_2} dt \int_D dx \frac{d}{dt} \Phi_t(x) \cdot \frac{d}{dt} \gamma_t(\Phi_t(x)) = 0, \tag{1.14}$$

where $\gamma_t = \gamma_t^0$ and γ_t^ε is defined by

$$\gamma_t^\varepsilon(\Phi_t(x)) = \frac{d}{d\varepsilon} \Phi_t^\varepsilon(x). \tag{1.15}$$

γ_t^ε is the vector field transversal to the motion that generates a flow pa-

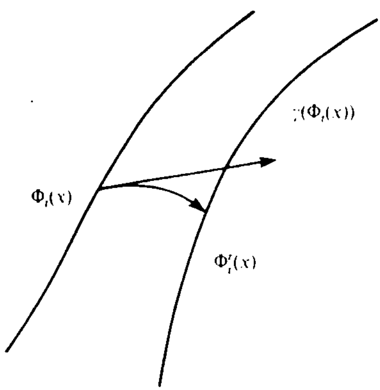

Figure 1.4

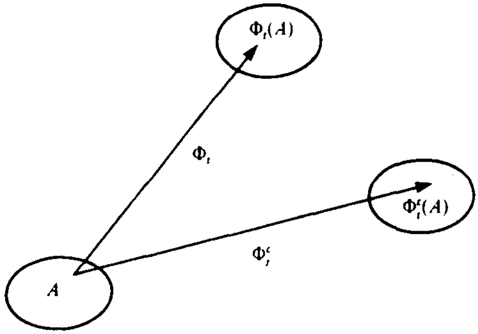

Figure 1.5

rametrized by ε (Fig. 1.4)

$$\Phi_t(x) \to \Phi_t^\varepsilon(x). \tag{1.16}$$

Obviously, such flow preserves the Lebesgue measure (Fig. 1.5)

$$(|\Phi_t^\varepsilon(A)| = |A| = |\Phi_t(A)|), \tag{1.17}$$

and hence, by the Liouville theorem,

$$\operatorname{div} \gamma_t = 0. \tag{1.18}$$

Moreover, it follows easily from definition (1.15) that

$$\gamma_t \cdot n = 0 \qquad \text{for} \quad x \in \partial D.$$

Coming back to (1.14), we obtain, by integration by parts,

$$\int_{t_1}^{t_2} dt \int_D dx \left\{ \frac{d^2}{dt^2} [\Phi_t(x)] \cdot \gamma_t(\Phi_t(x)) \right\} = 0. \tag{1.19}$$

Moreover,

$$\frac{d^2}{dt^2} \Phi_t(x) = \frac{d}{dt} u(\Phi_t(x), t) = D_t u(\Phi_t(x), t). \tag{1.20}$$

Here we used the notation

$$D_t f \equiv \partial_t f + (u \cdot \nabla) f = \partial_t f + \sum_{i=1}^{3} u_i \frac{\partial}{\partial x_i} f \tag{1.21}$$

for the derivative of a function f along the trajectories $\Phi_t(x)$ (D_t is sometimes also called the *material* or *substantial* or *molecular* derivative).

We insert (1.20) in (1.19). Since the Jacobian of the time transformation is one, by virtue of the arbitrariness of the times t_1 and t_2, we obtain

$$\int_D (D_t u)(x) \cdot \gamma_t(x) \, dx = 0. \tag{1.22}$$

From (1.22) it follows that $D_t u$ is orthogonal (in the sense of $L_2(D)$) to all divergence-free vector fields tangent to the border. (The arbitrariness of γ follows from the arbitrariness of Φ^ε). By virtue of a classical lemma (see Appendix 1.2), which states that a vector field, which is orthogonal to all the divergence-free fields tangent to the boundary, is the gradient of a scalar function, we can conclude that

$$D_t u = -\nabla p \tag{1.23}$$

for some function $p: R \times D \to R$. We observe that the minus sign in (1.23) is purely conventional.

Equation (1.23), together with the equations,

$$\nabla \cdot u = 0, \tag{1.24}$$

$$u \cdot n = 0 \quad \text{on } \partial D, \tag{1.25}$$

form the *Euler equation* for an ideal (or perfect) incompressible fluid.

The physical meaning of these equations is transparent: $D_t u$, the acceleration of a fluid particle, is equal to a force $-\nabla p$ to be determined on the basis of the principle of the incompressibility. $-\nabla p$ plays the same role as the constraint force for a free particle system constrained to move on a manifold. It is easy to verify (see Exercise 4) that a completely free motion in general violates the incompressibility condition. The scalar field $p = p(x, t)$ is called *pressure*.

An interesting class of solutions of the Euler equation are the *steady or stationary flows* which are the solutions, $u = u(x)$, not explicitly depending on time. For such flows the material derivative $D_t u$ consists only of the term $(u \cdot \nabla)u$, so that the stationary flows are those divergence-free fields for which $(u \cdot \nabla)u$ is the gradient of a scalar field. In this case, the integral lines of the velocity field are constant in time and they coincide with the trajectories of the particles of the fluid.

Equations (1.23), (1.24), (1.25) form a system of partial differential equations that we rewrite explicitly

$$\begin{gathered}
\partial_t u_j(x, t) + \sum_{i=1}^{3} [u_i \cdot \partial_i] u_j(x, t) = -\partial_j p(x, t), \\
\sum_{i=1}^{3} \partial_{x_j} u_j(x, t) = 0, \\
\sum_{i=1}^{3} u_i \cdot n_i(x) = 0.
\end{gathered} \tag{1.26}$$

This system of equations, in spite of the simplicity of the physical model from which they have been deduced, gives rise to a rather complicated mathematical problem, as we will see in detail in the next chapter. Here we want to outline only that the main problem of fluid dynamics consists in determining the velocity field, $u = u(x, t)$, at time t once known at time zero. When the velocity field is determined, the trajectories of the fluid particles are the