THE TOPOLOGY OF FIBRE BUNDLES

By Norman Steenrod

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Preface

The recognition of the domain of mathematics called fibre bundles took place in the period 1935–1940. The first general definitions were given by H. Whitney. His work and that of H. Hopf and E. Stiefel demonstrated the importance of the subject for the applications of topology to differential geometry. Since then, some seventy odd papers dealing with bundles have appeared. The subject has attracted general interest, for it contains some of the finest applications of topology to other fields, and gives promise of many more. It also marks a return of algebraic topology to its origin; and, after many years of introspective development, a revitalization of the subject from its roots in the study of classical manifolds.

No exposition of fibre bundles has appeared. The literature is in a state of partial confusion, due mainly to the experimentation with a variety of definitions of "fibre bundle." It has not been clear that any one definition would suffice for all results. The derivations of analogous conclusions from differing hypotheses have produced much overlapping. Many "known" results have not been published. It has been realized that certain standard theorems of topology are special cases of propositions about bundles, but the generalized forms have not been given.

The present treatment is an initial attempt at an organization. It grew out of lectures which I gave at the University of Michigan in 1947, and at Princeton University in 1948. The informed reader will find little here that is essentially new. Only such improvements and fresh applications are made as must accompany any reasonably successful organization.

The book is divided into three parts according to the demands made on the reader's knowledge of topology. The first part presupposes only a minimum of point set theory and closes with two articles dealing with covering spaces and the fundamental group. Part II makes extensive use of the homotopy groups of Hurewicz. Since no treatment of these has appeared in book form, Part II opens with a survey of the subject. Definitions and results are stated in detail; some proofs are given, and others are indicated. In Part III we make use of cohomology theory. Here, again, a survey is required because the standard treatments do not include the generalized form we must use. A reader who is familiar with the elements of homology theory will have little difficulty.

I must acknowledge my gratitude to Professor Sze-tsen Hu and Dr. R. L. Taylor who read the manuscript and suggested many improvements.

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Numbers enclosed in brackets refer to the bibliography.

NORMAN STEENROD

May, 1950
Princeton University

Contents

Part I. The General Theory of Bundles

Part II. The Homotopy Theory of Bundles

Part III. The Cohomology Theory of Bundles

THE TOPOLOGY OF FIBRE BUNDLES

Part I. The General Theory of Bundles

§1. Introduction

1.1. Provisional definition. A fibre bundle $\mathcal{B}$ consists, at least, of the following: (i) a topological space B called the *bundle space* (or, simply, bundle), (ii) a topological space X called the *base space,* (iii) a continuous map

$$p\colon \quad B \to X$$

of B onto X called the *projection,* and (iv) a space Y called the *fibre.* The set Y_x, defined by

$$Y_x = p^{-1}(x),$$

is called the *fibre over the point x of X.* It is required that each Y_x be homeomorphic to Y. Finally, for each x of X, there is a neighborhood V of x and a homeomorphism

$$\phi\colon \quad V \times Y \to p^{-1}(V)$$

such that

$$p\phi(x',y) = x' \qquad\qquad x' \in V,\ y \in Y.$$

A *cross-section* of a bundle is a continuous map $f\colon \ X \to B$ such that $pf(x) = x$ for each $x \in X$.

The above definition of bundle is not sufficiently restrictive. A bundle will be required to carry additional structure involving a group G of homeomorphisms of Y called the *group of the bundle.* Before imposing the additional requirements, consideration of a collection of examples will show the need for these. The discussion of these examples will be brief and intuitive; each will be treated later in detail.

1.2. The product bundle. The first example is the *product bundle* or product space $B = X \times Y$. In this case, the projection is given by $p(x,y) = x$. Taking $V = X$ and $\phi =$ the identity, the last condition is fulfilled. The cross-sections of B are just the graphs of maps $X \to Y$. The fibres are, of course, all homeomorphic, however there is a natural unique homeomorphism $Y_x \to Y$ given by $(x,y) \to y$. As will be seen, this is equivalent to the statement that the group G of the bundle consists of the identity alone.

1.3. The Möbius band. The second example is the Möbius band. The base space X is a circle obtained from a line segment L (as indicated in Fig. 1) by identifying its ends. The fibre Y is a line segment. The bundle B is obtained from the product $L \times Y$ by matching the two ends with a twist. The projection $L \times Y \to L$ carries over under this matching into a projection p: $B \to X$. There are numerous cross-sections; any curve as indicated with end points that match provides a cross-section. It is clear that any two cross-sections must agree on at least one point. There is no natural unique homeomorphism of Y_x

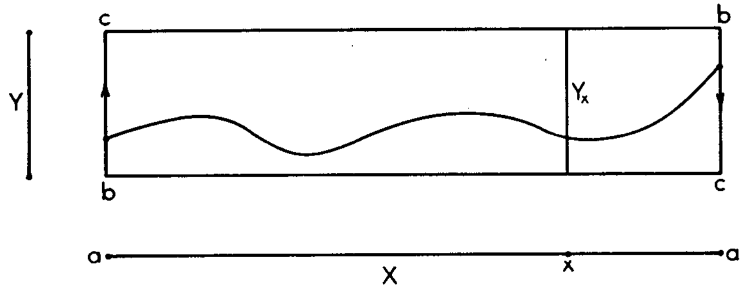

FIG. 1.

with Y. However there are two such which differ by the map g of Y on itself obtained by reflecting in its midpoint. In this case the group G is the cyclic group of order 2 generated by g.

1.4. The Klein bottle. The third example is the Klein bottle. The preceding construction is modified by replacing the fibre by a circle (Fig. 2). The ends of the cylinder $L \times Y$ are identified, as indicated, by reflecting in the diameter de. Again, the group G, is the cyclic group

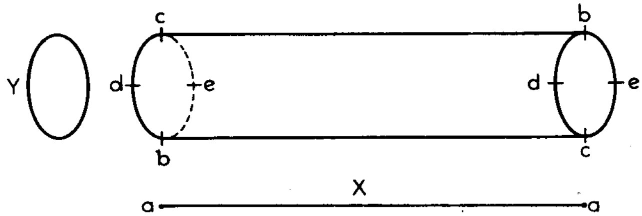

FIG. 2.

of order 2 generated by this reflection. (It is impossible to visualize this example in complete detail since the Klein bottle cannot be imbedded topologically in euclidean 3-space.)

1.5. The twisted torus. The fourth example, we will call the *twisted torus*. The construction is the same as for the Klein bottle except that reflection in the diameter de is replaced by reflection in the center of the circle (or rotation through 180°). As before, the group G is cyclic of order 2. In contrast to the preceding two examples, this bundle is homeomorphic to the product space $X \times Y$ and in such a way as to preserve fibres. However to achieve this one must use homeomorphisms $Y \to Y_x$ other than the two natural ones. But they need not differ from these by more than rotations of Y. This behavior is expressed by saying that the twisted torus is not a product bundle, but it is equivalent to one in the full group of rotations of Y.

1.6. Covering spaces. A covering space B of a space X is another example of a bundle. The projection p: $B \to X$ is the covering map. The usual definition of a covering space is the definition of bundle, in §1.1, modified by requiring that each Y_x is a discrete subspace of B, and that ϕ is a homeomorphism of $V \times Y_x$ with $p^{-1}(V)$ so that $\phi(x,y) = y$. If, in addition, it is supposed that X is arcwise connected, motion of a point x along a curve C in X from x_1 to x_2 can be covered by a continuous motion of Y_x in B from Y_{x_1} to Y_{x_2}. Choosing a base point x_0, each Y_x can be put in 1-1 correspondence with $Y = Y_{x_0}$ using a curve in X. This correspondence depends only on the homotopy class of the curve. Considering the action on Y of closed curves from x_0 to x_0, the fundamental group $\pi_1(X)$ appears as a group of permutations on Y. Any two correspondences of Y_x with Y differ by a permutation corresponding to an element of $\pi_1(X)$. Thus, *for covering spaces, the group of the bundle is a factor group of the fundamental group of the base space.*

1.7. Coset spaces. Another example of a bundle is a Lie group B operating as a transitive group of transformations on a manifold X. The projection is defined by selecting a point $x_0 \varepsilon X$ and defining $p(b) = b(x_0)$. If Y is the subgroup of B which leaves x_0 fixed, then the fibres are just the left cosets of Y in B. There are many natural correspondences $Y \to Y_x$, any $b \varepsilon Y_x$ defines one by $y \to b\cdot y$. However any two such $y \to b\cdot y$, $y \to b'\cdot y$ differ by the left translation of Y corresponding to $b^{-1}b'$. Thus the group G of the bundle coincides with the fibre Y and acts on Y by left translations. Finding a cross-section for such a bundle is just the problem of constructing in B a simply-transitive continuous family of transformations.

1.8. The tangent bundle of a manifold. As a final example let X be an n-dimensional differentiable manifold, let B be the set of all tangent vectors at all points of X, and let p assign to each vector its initial point. Then Y_x is the tangent plane at x. It is a linear space. Choosing a single representative Y, linear correspondences $Y_x \to Y$ can be constructed (using chains of coordinate neighborhoods in X), but not uniquely. In this case the group G of the bundle is the full linear group operating on Y. A cross-section here is just a vector field over X. The entire bundle is called the tangent bundle of X.

1.9. Generalizations of product spaces. It is to be observed that all the preceding examples of bundles are very much like product spaces. The language and notation has been designed to reflect this fact. A bundle is a generalization of a product space. The study of two spaces X and Y and maps f: $X \to Y$ is equivalent to the study of the product space $X \times Y$, its projections into X and Y, and graphs of maps f. This is broadened by replacing $X \times Y$ by a bundle space B, sacrificing the projection into Y, but replacing it, for each x, by a family of maps $Y_x \to Y$ any two of which differ by an element of a group G operating on Y. The graphs of continuous functions f: $X \to Y$ are replaced by cross-sections of the bundle.

This point of view would lead one to expect that most of the concepts of topology connected with pairs of spaces and their maps should generalize in some form. This is sustained in all that follows. For example, the Hopf theorem on the classification of maps of an n-complex into an n-sphere generalizes into the theory of the characteristic cohomology classes of a sphere-bundle.

The problems connected with bundles are of various types. The simplest question is the one of existence of a cross-section. This is of importance in differential geometry where a tensor field with prescribed algebraic properties is to be constructed. Is the bundle equivalent to a product bundle? If so, there exist many cross-sections. What are the relations connecting the homology and homotopy groups of the base space, bundle, fibre, and group? Can the bundle be simplified by replacing the group G by a smaller one? For given X, Y, G, what are the possible distinct bundles B? This last is the classification problem.

§2. COORDINATE BUNDLES AND FIBRE BUNDLES

2.1. The examples of §1 show that a bundle carries, as part of its structure, a group G of transformations of the fibre Y. In the last two examples, the group G has a topology. It is necessary to weave G and its topology into the definition of the bundle. This will be achieved through the intermediate notion of a *fibre bundle with coordinate systems* (briefly: "coordinate bundle"). The coordinate systems are eliminated by a notion of equivalence of coordinate bundles, and a passage to equivalence classes.

2.2. Transformation groups. A *topological group* G is a set which has a group structure and a topology such that (a) g^{-1} is continuous for g in G, and (b) g_1g_2 is continuous simultaneously in g_1 and g_2, i.e. the map $G \times G \to G$ given by $(g_1,g_2) \to g_1g_2$ is continuous when $G \times G$ has the usual topology of a product space.

If G is a topological group, and Y is a topological space, we say that G *is a topological transformation group of* Y *relative to a map* η: $G \times Y \to Y$ if (i) η is continuous, (ii) $\eta(e,y) = y$ where e is the identity of G, and (iii) $\eta(g_1g_2,y) = \eta(g_1,\eta(g_2,y))$ for all g_1,g_2 in G and y in Y.

As we shall rarely consider more than one such η, we shall abbreviate $\eta(g,y)$ by $g\cdot y$. Then (ii) becomes $e\cdot y = y$ and (iii) becomes $(g_1g_2)\cdot y = g_1\cdot(g_2\cdot y)$. For any fixed g, $y \to g\cdot y$ is a homeomorphism of Y onto itself; for it has the continuous inverse $y \to g^{-1}\cdot y$. In this way η provides a homomorphism of G into the group of homeomorphisms of Y.

We shall say that G is *effective* if $g\cdot y = y$, for all y, implies $g = e$. Then G is isomorphic to a group of homeomorphisms of Y. In this case one might identify G with the group of homeomorphisms, however we shall frequently allow the same G to operate on several spaces.

Unless otherwise stated, a topological transformation group will be assumed to be effective.

2.3. Definition of coordinate bundle. A *coordinate bundle* $\mathcal{B}$ is a collection as follows:

(1) A space B called the *bundle space*,
(2) a space X called the *base space*,
(3) a map p: $B \to X$ of B onto X called the *projection*,
(4) a space Y called the *fibre*,
(5) an effective topological transformation group G of Y called the *group of the bundle*,
(6) a family $\{V_j\}$ of open sets covering X indexed by a set J, the V_j's are called *coordinate neighborhoods*, and
(7) for each j in J, a homeomorphism

$$\phi_j\colon\ V_j \times Y \to p^{-1}(V_j)$$

called the *coordinate function*.

The coordinate functions are required to satisfy the following conditions:

(8) $$p\phi_j(x,y) = x, \qquad \text{for } x \in V_j,\ y \in Y,$$

(9) if the map $\phi_{j,x}$: $Y \to p^{-1}(x)$ is defined by setting

$$\phi_{j,x}(y) = \phi_j(x,y),$$

then, for each pair i,j in J, and each $x \in V_i \cap V_j$, the homeomorphism

$$\phi_{j,x}^{-1}\phi_{i,x}\colon\ Y \to Y$$

coincides with the operation of an element of G (it is unique since G is effective), and

(10) for each pair i,j in J, the map

$$g_{ji}\colon \quad V_i \cap V_j \rightarrow G$$

defined by $g_{ji}(x) = \phi_{j,x}^{-1}\phi_{i,x}$ is continuous.

It is to be observed that without (5), (9) and (10) the notion of bundle would be just that of §1.1. The condition (9) ties G essentially into the structure of the bundle, and (10) does the same for the topology of G.

As in §1, we denote $p^{-1}(x)$ by Y_x and call it the *fibre over* x.

The functions g_{ji} defined in (10) are called the *coordinate transformations* of the bundle. An immediate consequence of the definition is that, for any i,j,k in J,

(11) $$g_{kj}(x)g_{ji}(x) = g_{ki}(x), \qquad x \in V_i \cap V_j \cap V_k.$$

If we specialize by setting $i = j = k$, then

(12) $$g_{ii}(x) = \text{identity of } G, \qquad x \in V_i.$$

Now set $i = k$ in (11) and apply (12) to obtain

(13) $$g_{jk}(x) = [g_{kj}(x)]^{-1}, \qquad x \in V_j \cap V_k.$$

It is convenient to introduce the map

(14) $$p_j\colon \quad p^{-1}(V_j) \rightarrow Y$$

defined by

$$p_j(b) = \phi_{j,x}^{-1}(b) \qquad \text{where } x = p(b).$$

Then p_j satisfies the identities

(14′) $$p_j\phi_j(x,y) = y, \qquad \phi_j(p(b),p_j(b)) = b,$$
$$g_{ji}(p(b))\cdot p_i(b) = p_j(b), \qquad p(b) \in V_i \cap V_j.$$

2.4. Definition of fibre bundle. Two coordinate bundles $\mathfrak{B}$ and $\mathfrak{B}'$ are said to be *equivalent in the strict sense* if they have the same bundle space, base space, projection, fibre, and group, and their coordinate functions $\{\phi_j\}$, $\{\phi'_k\}$ satisfy the conditions that

(15) $$\bar{g}_{kj}(x) = \phi'^{-1}_{k,x}\phi_{j,x}, \qquad x \in V_j \cap V'_k$$

coincides with the operation of an element of G, and the map

$$\bar{g}_{kj}\colon \quad V_j \cap V'_k \rightarrow G$$

so obtained is continuous.

This can be stated briefly by saying that the union of the two sets of coordinate functions is a set of coordinate functions of a bundle.

That this is a proper equivalence relation follows quickly. Reflexivity is immediate. Symmetry follows from the continuity of $g \to g^{-1}$. Transitivity depends on the simultaneous continuity of $(g_1,g_2) \to g_1g_2$.

With this notion of equivalence, a *fibre bundle* is defined to be an equivalence class of coordinate bundles.

One may regard a fibre bundle as a "maximal" coordinate bundle having all possible coordinate functions of an equivalence class. As our indexing sets are unrestricted, this involves the usual logical difficulty connected with the use of the word "all."

2.5. Mappings of bundles. Let $\mathcal{B}$ and $\mathcal{B}'$ be two coordinate bundles having the same fibre and the same group. By a *map* h: $\mathcal{B} \to \mathcal{B}'$ is meant a continuous map h: $B \to B'$ having the following properties

(16) h carries each fibre Y_x of B homeomorphically onto a fibre $Y_{x'}$ of B', thus inducing a continuous map $\bar{h}$: $X \to X'$ such that

$$p'h = \bar{h}p,$$

(17) if $x \in V_j \cap \bar{h}^{-1}(V'_k)$, and h_x: $Y_x \to Y_{x'}$ is the map induced by h $(x' = \bar{h}(x))$, then the map

$$\bar{g}_{kj}(x) = \phi'^{-1}_{k,x'}h_x\phi_{j,x} = p'_kh_x\phi_{j,x}$$

of Y into Y coincides with the operation of an element of G, and

(18) the map

$$\bar{g}_{kj}: \quad V_j \cap \bar{h}^{-1}(V'_k) \to G$$

so obtained is continuous.

In the literature, the map h is called "fibre preserving." We shall use frequently the expression "bundle map" to emphasize that h is a map in the above sense.

It is readily proved that the identity map $B \to B$ is a map $\mathcal{B} \to \mathcal{B}$ in this sense. Likewise the composition of two maps $\mathcal{B} \to \mathcal{B}' \to \mathcal{B}''$ is also a map $\mathcal{B} \to \mathcal{B}''$.

A map of frequent occurrence is an inclusion map $\mathcal{B} \subset \mathcal{B}'$ obtained as follows. Let $\mathcal{B}'$ be a coordinate bundle over X', and let X be a subspace of X'. Let $B = p'^{-1}(X)$, $p = p'|B$, and define the coordinate functions of $\mathcal{B}$ by $\phi_j = \phi'_j|(V'_j \cap X) \times Y$. Then $\mathcal{B}$ is a coordinate bundle, and the inclusion map $B \to B'$ is a map $\mathcal{B} \to \mathcal{B}'$. We call $\mathcal{B}$ *the portion of* $\mathcal{B}'$ *over* X (or $\mathcal{B}$ is $\mathcal{B}'$ restricted to X), and we will use the notations

$$\mathcal{B} = \mathcal{B}'|X = \mathcal{B}'_X.$$

The functions $\bar{g}_{kj}$ of (17) and (18) are called the *mapping transformations*. There are two sets of relations which they satisfy:

$$\begin{aligned} \bar{g}_{kj}(x)g_{ji}(x) &= \bar{g}_{ki}(x), \quad x \in V_i \cap V_j \cap \bar{h}^{-1}(V'_k), \\ g'_{lk}(\bar{h}(x))\bar{g}_{kj}(x) &= \bar{g}_{lj}(x), \quad x \in V_j \cap \bar{h}^{-1}(V'_k \cap V'_l). \end{aligned} \tag{19}$$

These are verified by direct substitution using the definitions (10) and (17).

2.6. Lemma. *Let $\mathcal{B}$, $\mathcal{B}'$ be coordinate bundles having the same fibre Y and group G, and let $\bar{h}$: $X \to X'$ be a map of one base space into the other. Finally, let $\bar{g}_{kj}$: $V_j \cap \bar{h}^{-1}(V'_k) \to G$ be a set of continuous maps satisfying the conditions (19). Then there exists one and only one map h: $\mathcal{B} \to \mathcal{B}'$ inducing $\bar{h}$ and having $\{\bar{g}_{jk}\}$ as its mapping transformations.*

If $p(b) = x$ lies in $V_j \cap \bar{h}^{-1}(V'_k)$, define

$$h_{kj}(b) = \phi'_k(\bar{h}(x), \bar{g}_{kj}(x) \cdot p_j(b)). \tag{20}$$

Then h_{kj} is continuous in b, and $p'h_{kj}(b) = \bar{h}(p(b))$. Suppose $x \in V_i \cap V_j \cap \bar{h}^{-1}(V'_k \cap V'_l)$. Using the relations (14′) and (19), we have (with $x' = \bar{h}(x)$)

$$\begin{aligned} h_{kj}(b) &= \phi'_k(x', \bar{g}_{kj}(x)g_{ji}(x) \cdot p_i(b)) \\ &= \phi'_k(x', \bar{g}_{ki}(x) \cdot p_i(b)) = h_{ki}(b) \\ &= \phi'_l(x', g'_{lk}(x')\bar{g}_{ki}(x) \cdot p_i(b)) \\ &= \phi'_l(x', \bar{g}_{li}(x) \cdot p_i(b)) = h_{li}(b). \end{aligned}$$

It follows that any two functions of the collection $\{h_{kj}\}$ agree on their common domain. Since their domains are open and cover B, they define a single-valued continuous function h. Then $p'h = \bar{h}p$ follows from the same relation for h_{jk}. If, in (20), we replace b by $\phi_{j,x}(y)$, apply p'_k to both sides, and use the relations (14′), we obtain

$$\begin{aligned} p'_k h \phi_{j,x}(y) &= p'_k \phi'_k(x', \bar{g}_{kj}(x) \cdot p_j \phi_{j,x}(y)) \\ &= \bar{g}_{kj}(x) \cdot y \end{aligned}$$

which shows that h has the prescribed mapping transformations.

Conversely any h which has the prescribed mapping transformations must satisfy (20), and therefore h is unique.

2.7. Lemma. *Let $\mathcal{B}$, $\mathcal{B}'$ be coordinate bundles having the same fibre and group, and let h: $\mathcal{B} \to \mathcal{B}'$ be a map such that the induced map $\bar{h}$: $X \to X'$ is 1-1 and has a continuous inverse $\bar{h}^{-1}$: $X' \to X$. Then h has a continuous inverse h^{-1}: $B' \to B$, and h^{-1} is a map $\mathcal{B}' \to \mathcal{B}$.*

The fact that h is 1-1 in the large is evident. For any x' in $V'_k \cap \bar{h}(V_j)$, let $x = \bar{h}^{-1}(x')$, and, following (17), define

$$\bar{g}_{jk}(x') = \phi_{j,x}^{-1} h_x^{-1} \phi'_{k,x'}.$$

It follows that $\bar{g}_{jk}(x') = \bar{g}_{kj}(x)^{-1}$. Since $g \to g^{-1}$ is continuous in G, x is continuous in x', and $\bar{g}_{kj}(x)$ is continuous in x, it follows that $\bar{g}_{jk}(x')$ is continuous in x'. If $p'(b') = x'$ is in $V'_k \cap h(V_j)$, then h^{-1} is given by

$$h^{-1}(b') = \phi_j(\bar{h}^{-1}(x'), \bar{g}_{jk}(x') \cdot p'_k(b'))$$

which shows that h^{-1} is continuous on $p'^{-1}(V'_k \cap h(V_j))$. Since these sets are open and cover B', it follows that h^{-1} is continuous, and the lemma is proved.

Two coordinate bundles $\mathfrak{B}$ and $\mathfrak{B}'$ having the same base space, fibre and group are said to be *equivalent* if there exists a map $\mathfrak{B} \to \mathfrak{B}'$ which induces the identity map of the common base space.

The symmetry of this relation is provided by the above lemma. The reflexivity and transitivity are immediate. It is to be noted that strict equivalence, defined in §2.4, implies equivalence.

Two fibre bundles (see §2.4) having the same base space, fibre and group are said to be *equivalent* if they have representative coordinate bundles which are equivalent.

It is possible to define broader notions of equivalences of fibre bundles by allowing X or (Y,G) to vary by a topological equivalence. The effect of this is to reduce the number of equivalence classes. The definition chosen is the one most suitable for the classification theorems proved later.

2.8. Lemma. *Let $\mathfrak{B}, \mathfrak{B}'$ be coordinate bundles having the same base space, fibre, and group, then they are equivalent if and only if there exist continuous maps*

$$\bar{g}_{kj}: \quad V_j \cap V'_k \to G \qquad j \in J, \; k \in J'$$

such that

$$\text{(19')} \qquad \begin{aligned} \bar{g}_{ki}(x) &= \bar{g}_{kj}(x) g_{ji}(x), & x \in V_i \cap V_j \cap V'_k \\ \bar{g}_{lj}(x) &= g'_{lk}(x) \bar{g}_{kj}(x), & x \in V_j \cap V'_k \cap V'_l. \end{aligned}$$

Suppose, first that $\mathfrak{B}, \mathfrak{B}'$ are equivalent and h: $\mathfrak{B} \to \mathfrak{B}'$. Define $\bar{g}_{kj}$ by (17) (note that $x' = x$ since $\bar{h}$ is the identity). The relations (19) reduce to (19′).

Conversely, suppose the $\bar{g}_{kj}$ are given. The relations (19′) imply (19) in the case $\bar{h}$ = identity. The existence of h is provided by 2.6.

2.9. Let $\mathfrak{B}$ be a coordinate bundle with neighborhoods $\{V_j\}$, and let $\{V'_k\}$ be a covering of X by an indexed family of open sets such that