C*-ALGEBRAS AND OPERATOR THEORY

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Preface

This is an introductory textbook to a vast subject, which although more than fifty years old is still extremely active and rapidly expanding, and coming to have an increasingly greater impact on other areas of mathematics, as well as having applications to theoretical physics. I have attempted to give a leisurely and accessible exposition of the core material of the subject, and to cover a number of topics (the theory of C*-tensor products and K-theory) having a high contemporary profile. There was no intention to be encyclopedic, and many important topics had to be omitted in order to keep to a moderate size.

This book is aimed at the beginning graduate student and the specialist in another area who wishes to know the basics of this subject. The reader is assumed to have a good background in real and complex analysis, point set topology, measure theory, and elementary general functional analysis. Thus, such results as the Hahn–Banach extension theorem, the uniform boundedness principle, the Stone–Weierstrass theorem, and the Riesz–Kakutani theorem are assumed known. However, the theory of locally convex spaces is not presupposed, and the relevant material including the Krein–Milman theorem and the separation theorem are developed in a brief appendix. The book is arranged so that the appendix is not used until Chapter 4, and the first three chapters can, if desired, form the basis of a short course. The background material for the book is covered by the following textbooks: [Coh], [Kel], [Rud 1], and [Rud 2].

Each chapter concludes with a list of exercises arranged roughly according to the order in which the relevant item appeared in the chapter, and statements of additional results related to, and extending, the material in the text.

The symbols **N**, **Z**, **R**, $\mathbf{R}^+$, and **C** refer, respectively, to the sets of non-negative integers, integers, real numbers, non-negative real numbers, and complex numbers. Other notation is explained as needed.

The reader who has finished this book and wants direction for further study may refer to the Notes section where some books are recommended.

I am indebted to many authors of books on operator theory and operator algebras. Section 7.5 of this book is based on the approach of J. Cuntz to K-theory. I should like to thank my colleagues Trevor West and Martin Mathieu for reading preliminary drafts of some of the earlier chapters.

Gerard J. Murphy

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Contents

CHAPTER 1

Elementary Spectral Theory

In this chapter we cover the basic results of spectral theory. The most important of these are the non-emptiness of the spectrum, Beurling's spectral radius formula, and the Gelfand representation theory for commutative Banach algebras. We also introduce compact and Fredholm operators and analyse their elementary theory. Important concepts here are the essential spectrum and the Fredholm index.

Throughout this book the ground field for all vector spaces and algebras is the complex field **C**, unless the contrary is explicitly indicated in a particular context.

1.1. Banach Algebras

We begin by setting up the basic vocabulary needed to discuss Banach algebras and by giving some examples.

An *algebra* is a vector space A together with a bilinear map

$$A^2 \to A, \quad (a, b) \mapsto ab,$$

such that

$$a(bc) = (ab)c \quad (a, b, c \in A).$$

A *subalgebra* of A is a vector subspace B such that $b, b' \in B \Rightarrow bb' \in B$. Endowed with the multiplication got by restriction, B is itself an algebra.

A norm $\|.\|$ on A is said to be *submultiplicative* if

$$\|ab\| \leq \|a\|\|b\| \quad (a, b \in A).$$

In this case the pair $(A, \|.\|)$ is called a *normed algebra*. If A admits a unit 1 ($a1 = 1a = a$, for all $a \in A$) and $\|1\| = 1$, we say that A is a *unital normed algebra*.

If A is a normed algebra, then it is evident from the inequality

$$\|ab - a'b'\| \leq \|a\|\|b - b'\| + \|a - a'\|\|b'\|$$

that the multiplication operation $(a, b) \mapsto ab$ is jointly continuous.

A complete normed algebra is called a *Banach algebra*. A complete unital normed algebra is called a *unital Banach algebra*.

A subalgebra of a normed algebra is obviously itself a normed algebra with the norm got by restriction. The closure of a subalgebra is a subalgebra. A closed subalgebra of a Banach algebra is a Banach algebra.

1.1.1. *Example.* If S is a set, $\ell^\infty(S)$, the set of all bounded complex-valued functions on S, is a unital Banach algebra where the operations are defined pointwise:

$$\begin{aligned}(f + g)(x) &= f(x) + g(x)\\ (fg)(x) &= f(x)g(x)\\ (\lambda f)(x) &= \lambda f(x),\end{aligned}$$

and the norm is the sup-norm

$$\|f\|_\infty = \sup_{x \in S} |f(x)|.$$

1.1.2. *Example.* If Ω is a topological space, the set $C_b(\Omega)$ of all bounded continuous complex-valued functions on Ω is a closed subalgebra of $\ell^\infty(\Omega)$. Thus, $C_b(\Omega)$ is a unital Banach algebra.

If Ω is compact, $C(\Omega)$, the set of continuous functions from Ω to $\mathbf{C}$, is of course equal to $C_b(\Omega)$.

1.1.3. *Example.* If Ω is a locally compact Hausdorff space, we say that a continuous function f from Ω to $\mathbf{C}$ *vanishes at infinity*, if for each positive number ε the set $\{\omega \in \Omega \mid |f(\omega)| \geq \varepsilon\}$ is compact. We denote the set of such functions by $C_0(\Omega)$. It is a closed subalgebra of $C_b(\Omega)$, and therefore, a Banach algebra. It is unital if and only if Ω is compact, and in this case $C_0(\Omega) = C(\Omega)$. The algebra $C_0(\Omega)$ is one of the most important examples of a Banach algebra, and we shall see it used constantly in C*-algebra theory (the functional calculus).

1.1.4. *Example.* If (Ω, μ) is a measure space, the set $L^\infty(\Omega, \mu)$ of (classes of) essentially bounded complex-valued measurable functions on Ω is a unital Banach algebra with the usual (pointwise-defined) operations and the essential supremum norm $f \mapsto \|f\|_\infty$.

1.1.5. *Example.* If Ω is a measurable space, let $B_\infty(\Omega)$ denote the set of all bounded complex-valued measurable functions on Ω. Then $B_\infty(\Omega)$ is a closed subalgebra of $\ell^\infty(\Omega)$, so it is a unital Banach algebra. This example will be used in connection with the spectral theorem in Chapter 2.

1.1.6. *Example.* The set A of all continuous functions on the closed unit disc $\mathbf{D}$ in the plane which are analytic on the interior of $\mathbf{D}$ is a closed subalgebra of $C(\mathbf{D})$, so A is a unital Banach algebra, called the *disc algebra*. This is the motivating example in the theory of function algebras, where many aspects of the theory of analytic functions are extended to a Banach algebraic setting.

All of the above examples are of course *abelian*—that is, $ab = ba$ for all elements a and b—but the following examples are not, in general.

1.1.7. *Example.* If X is a normed vector space, denote by $B(X)$ the set of all bounded linear maps from X to itself (the *operators* on X). It is routine to show that $B(X)$ is a normed algebra with the pointwise-defined operations for addition and scalar multiplication, multiplication given by $(u, v) \mapsto u \circ v$, and norm the *operator norm*:

$$\|u\| = \sup_{x \neq 0} \frac{\|u(x)\|}{\|x\|} = \sup_{\|x\| \leq 1} \|u(x)\|.$$

If X is a Banach space, $B(X)$ is complete and is therefore a Banach algebra.

1.1.8. *Example.* The algebra $M_n(\mathbf{C})$ of $n \times n$-matrices with entries in $\mathbf{C}$ is identified with $B(\mathbf{C}^n)$. It is therefore a unital Banach algebra. Recall that an *upper triangular* matrix is one of the form

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} & \dots & \dots & \lambda_{1n} \\ 0 & \lambda_{22} & \dots & \dots & \lambda_{2n} \\ 0 & 0 & \lambda_{33} & \dots & \lambda_{3n} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \lambda_{nn} \end{pmatrix}$$

(all entries below the main diagonal are zero). These matrices form a subalgebra of $M_n(\mathbf{C})$.

We shall be seeing many more examples of Banach algebras as we proceed. Most often these will be non-abelian, but in the first three sections of this chapter we shall be principally concerned with the abelian case.

If $(B_\lambda)_{\lambda \in \Lambda}$ is a family of subalgebras of an algebra A, then $\cap_{\lambda \in \Lambda} B_\lambda$ is a subalgebra, also. Hence, for any subset S of A, there is a smallest subalgebra B of A containing S (namely, the intersection of all the subalgebras containing S). This algebra is called the subalgebra of A *generated* by S. If S is the singleton set $\{a\}$, then B is the linear span of all powers a^n ($n = 1, 2, \ldots$) of a. If A is a normed algebra, the closed algebra C *generated* by a set S is the smallest closed subalgebra containing S. It is plain that $C = \bar{B}$, where B is the subalgebra generated by S.

If $A = C(\mathbf{T})$, where $\mathbf{T}$ is the unit circle, and if $z: \mathbf{T} \to \mathbf{C}$ is the inclusion function, then the closed algebra generated by z and its conjugate $\bar{z}$ is $C(\mathbf{T})$ itself (immediate from the Stone–Weierstrass theorem).

A *left* (respectively, *right*) *ideal* in an algebra A is a vector subspace I of A such that

$$a \in A \quad \text{and} \quad b \in I \Rightarrow ab \in I \quad (\text{respectively, } ba \in I).$$

An *ideal* in A is a vector subspace that is simultaneously a left and a right ideal in A. Obviously, 0 and A are ideals in A, called the *trivial* ideals. A *maximal* ideal in A is a proper ideal (that is, it is not A) that is not contained in any other proper ideal in A. Maximal left ideals are defined similarly.

An ideal I is *modular* if there is an element u in A such that $a - au$ and $a - ua$ are in I for all $a \in A$. It follows easily from Zorn's lemma that every proper modular ideal is contained in a maximal ideal.

If ω is an element of a locally compact Hausdorff space Ω, and $M_\omega = \{f \in C_0(\Omega) \mid f(\omega) = 0\}$, then M_ω is a modular ideal in the algebra $C_0(\Omega)$. This is so because there is an element $u \in C_0(\Omega)$ such that $u(\omega) = 1$, and hence, $f - uf \in M_\omega$ for all $f \in C_0(\Omega)$. Since M_ω is of codimension one in $C_0(\Omega)$ (as $M \oplus \mathbf{C}u = C_0(\Omega)$), it is a maximal ideal.

If I is an ideal of A, then A/I is an algebra with the multiplication given by

$$(a + I)(b + I) = ab + I.$$

If I is modular, then A/I is unital (if $a - au, a - ua \in I$ for all $a \in A$, then $u + I$ is the unit). Conversely, if A/I is unital then I is modular.

If A is unital, then obviously all its ideals are modular, and therefore, A posesses maximal ideals.

If $(I_\lambda)_{\lambda \in \Lambda}$ is a family of ideals of an algebra A, then $\cap_{\lambda \in \Lambda} I_\lambda$ is an ideal of A. Hence, if $S \subseteq A$, there is a smallest ideal I of A containing S. We call I the ideal *generated* by S. If A is a normed algebra, then the closure of an ideal is an ideal. The closed ideal J *generated* by a set S is the smallest closed ideal containing S. It is clear that J is the closure of the ideal generated by S.

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1.1.1. Theorem. *If I is a closed ideal in a normed algebra A, then A/I is a normed algebra when endowed with the quotient norm*

$$\|a+I\| = \inf_{b\in I} \|a+b\|.$$

Proof. Let $\varepsilon > 0$ and suppose that a, b belong to A. Then $\varepsilon + \|a+I\| > \|a+a'\|$ and $\varepsilon + \|b+I\| > \|b+b'\|$ for some $a', b' \in I$. Hence,

$$(\varepsilon + \|a+I\|)(\varepsilon + \|b+I\|) > \|a+a'\|\|b+b'\| \geq \|ab+c\|,$$

where $c = a'b + ab' + a'b' \in I$. Thus, $(\varepsilon + \|a+I\|)(\varepsilon + \|b+I\|) \geq \|ab+I\|$. Letting $\varepsilon \to 0$, we get $\|a+I\|\|b+I\| \geq \|ab+I\|$; that is, the quotient norm is submultiplicative. □

A *homomorphism* from an algebra A to an algebra B is a linear map $\varphi: A \to B$ such that $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in A$. Its kernel $\ker(\varphi)$ is an ideal in A and its image $\varphi(A)$ is a subalgebra of B. We say φ is *unital* if A and B are unital and $\varphi(1) = 1$.

If I is an ideal in A, the quotient map $\pi: A \to A/I$ is a homomorphism.

If φ, ψ are continuous homomorphisms from a normed algebra A to a normed algebra B, then $\varphi = \psi$ if φ and ψ are equal on a set S that generates A as a normed algebra (that is, A is the closed algebra generated by S). This follows from the observation that the set $\{a \in A \mid \varphi(a) = \psi(a)\}$ is a closed subalgebra of A.

If A is the disc algebra and $\lambda \in \mathbf{D}$, the function

$$A \to \mathbf{C}, \quad f \mapsto f(\lambda),$$

is a continuous homomorphism. Moreover, every non-zero continuous homomorphism from A to $\mathbf{C}$ is of this form. This follows from the fact that the closed subalgebra generated by the unit and the inclusion function $z: \mathbf{D} \to \mathbf{C}$ is A. We show this: If $f \in A$ and $0 < r < 1$, define $f_r \in C(\mathbf{D})$ by $f_r(\lambda) = f(r\lambda)$. By uniform continuity of f on $\mathbf{D}$, we have $\lim_{r\to 1^-} \|f - f_r\|_\infty = 0$. Since f_r is extendable to an analytic function on the open disc of center 0 and radius $1/r$, it is the uniform limit on $\mathbf{D}$ of its Taylor series. Thus, f_r is the uniform limit of polynomial functions on $\mathbf{D}$, and therefore, so is f.

1.2. The Spectrum and the Spectral Radius

Let $\mathbf{C}[z]$ denote the algebra of all polynomials in an indeterminate z with complex coefficients. If a is an element of a unital algebra A and $p \in \mathbf{C}[z]$ is the polynomial

$$p = \lambda_0 + \lambda_1 z^1 + \cdots + \lambda_n z^n,$$

we set

$$p(a) = \lambda_0 1 + \lambda_1 a^1 + \cdots + \lambda_n a^n.$$

The map

$$\mathbf{C}[z] \to A, \quad p \mapsto p(a),$$

is a unital homomorphism.

We say that $a \in A$ is *invertible* if there is an element b in A such that $ab = ba = 1$. In this case b is unique and written a^{-1}. The set

$$\mathrm{Inv}(A) = \{a \in A \mid a \text{ is invertible}\}$$

is a group under multiplication.

We define the *spectrum* of an element a to be the set

$$\sigma(a) = \sigma_A(a) = \{\lambda \in \mathbf{C} \mid \lambda 1 - a \notin \mathrm{Inv}(A)\}.$$

We shall henceforth find it convenient to write $\lambda 1$ simply as λ.

1.2.1. ***Example.*** Let $A = C(\Omega)$, where Ω is a compact Hausdorff space. Then $\sigma(f) = f(\Omega)$ for all $f \in A$.

1.2.2. ***Example.*** Let $A = \ell^\infty(S)$, where S is a non-empty set. Then $\sigma(f) = (f(S))^-$ (the closure in $\mathbf{C}$) for all $f \in A$.

1.2.3. ***Example.*** Let A be the algebra of upper triangular $n \times n$-matrices. If $a \in A$, say

$$a = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1n} \\ 0 & \lambda_{22} & \dots & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_{nn} \end{pmatrix}$$

it is elementary that

$$\sigma(a) = \{\lambda_{11}, \lambda_{22}, \dots, \lambda_{nn}\}.$$

Similarly, if $A = M_n(\mathbf{C})$ and $a \in A$, then $\sigma(a)$ is the set of eigenvalues of a.

Thus, one thinks of the spectrum as simultaneously a generalisation of the range of a function and the set of eigenvalues of a finite square matrix.

1.2.1. ***Remark.*** If a, b are elements of a unital algebra A, then $1 - ab$ is invertible if and only if $1 - ba$ is invertible. This follows from the observation that if $1 - ab$ has inverse c, then $1 - ba$ has inverse $1 + bca$.

A consequence of this equivalence is that $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$ for all $a, b \in A$.

1.2.1. Theorem. *Let a be an element of a unital algebra A. If $\sigma(a)$ is non-empty and $p \in \mathbf{C}[z]$, then*

$$\sigma(p(a)) = p(\sigma(a)).$$

Proof. We may suppose that p is not constant. If $\mu \in \mathbf{C}$, there are elements $\lambda_0, \ldots, \lambda_n$ in $\mathbf{C}$, where $\lambda_0 \neq 0$, such that

$$p - \mu = \lambda_0(z - \lambda_1)\ldots(z - \lambda_n),$$

and therefore,

$$p(a) - \mu = \lambda_0(a - \lambda_1)\ldots(a - \lambda_n).$$

It is clear that $p(a) - \mu$ is invertible if and only if $a - \lambda_1, \ldots, a - \lambda_n$ are. It follows that $\mu \in \sigma(p(a))$ if and only if $\mu = p(\lambda)$ for some $\lambda \in \sigma(a)$, and therefore, $\sigma(p(a)) = p(\sigma(a))$. □

The spectral mapping property for polynomials is generalised to continuous functions in Chapter 2, but only for certain elements in certain algebras. There is a version of Theorem 1.2.1 for analytic functions and Banach algebras (see [Tak, Proposition 2.8], for example). We shall not need this, however.

1.2.2. Theorem. *Let A be a unital Banach algebra and a an element of A such that $\|a\| < 1$. Then $1 - a \in \mathrm{Inv}(A)$ and*

$$(1 - a)^{-1} = \sum_{n=0}^{\infty} a^n.$$

Proof. Since $\sum_{n=0}^{\infty} \|a^n\| \leq \sum_{n=0}^{\infty} \|a\|^n = (1 - \|a\|)^{-1} < +\infty$, the series $\sum_{n=0}^{\infty} a^n$ is convergent, to b say, in A, and since $(1 - a)(1 + \cdots + a^n) = 1 - a^{n+1}$ converges to $(1 - a)b = b(1 - a)$ and to 1 as $n \to \infty$, the element b is the inverse of $1 - a$. □

The series in Theorem 1.2.2 is called the *Neumann* series for $(1 - a)^{-1}$.

1.2.3. Theorem. *If A is a unital Banach algebra, then* $\mathrm{Inv}(A)$ *is open in A, and the map*

$$\mathrm{Inv}(A) \to A, \quad a \mapsto a^{-1},$$

is differentiable.

Proof. Suppose that $a \in \mathrm{Inv}(A)$ and $\|b - a\| < \|a^{-1}\|^{-1}$. Then $\|ba^{-1} - 1\| \leq \|b - a\|\|a^{-1}\| < 1$, so $ba^{-1} \in \mathrm{Inv}(A)$, and therefore, $b \in \mathrm{Inv}(A)$. Thus, $\mathrm{Inv}(A)$ is open in A.

If $b \in A$ and $\|b\| < 1$, then $1 + b \in \mathrm{Inv}(A)$ and

$$\|(1+b)^{-1} - 1 + b\| = \|\sum_{n=0}^{\infty}(-1)^n b^n - 1 + b\| = \|\sum_{n=2}^{\infty}(-1)^n b^n\|$$
$$\leq \sum_{n=2}^{\infty} \|b\|^n = \|b\|^2/(1 - \|b\|)^{-1}.$$

Let $a \in \mathrm{Inv}(A)$ and suppose that $\|c\| < \frac{1}{2}\|a^{-1}\|^{-1}$. Then $\|a^{-1}c\| < 1/2 < 1$, so (with $b = a^{-1}c$),

$$\|(1 + a^{-1}c)^{-1} - 1 + a^{-1}c\| \leq \|a^{-1}c\|^2/(1 - \|a^{-1}c\|)^{-1} \leq 2\|a^{-1}c\|^2,$$

since $1 - \|a^{-1}c\| > 1/2$. Now define u to be the linear operator on A given by $u(b) = -a^{-1}ba^{-1}$. Then,

$$\|(a+c)^{-1} - a^{-1} - u(c)\| = \|(1 + a^{-1}c)^{-1}a^{-1} - a^{-1} + a^{-1}ca^{-1}\|$$
$$\leq \|(1 + a^{-1}c)^{-1} - 1 + a^{-1}c\|\|a^{-1}\| \leq 2(\|a^{-1}\|^3\|c\|^2).$$

Consequently,

$$\lim_{c \to 0} \frac{\|(a+c)^{-1} - a^{-1} - u(c)\|}{\|c\|} = 0,$$

and therefore, the map $\sigma: b \mapsto b^{-1}$ is differentiable at $b = a$ with derivative $\sigma'(a) = u$. □

The algebra $\mathbf{C}[z]$ is a normed algebra where the norm is defined by setting

$$\|p\| = \sup_{|\lambda| \leq 1} |p(\lambda)|.$$

Observe that $\mathrm{Inv}(\mathbf{C}[z]) = \mathbf{C} \setminus \{0\}$, so the polynomials $p_n = 1 + z/n$ are not invertible. But $\lim_{n\to\infty} p_n = 1$, which shows that $\mathrm{Inv}(\mathbf{C}[z])$ is not open in $\mathbf{C}[z]$. Thus, the norm on $\mathbf{C}[z]$ is not complete.

1.2.4. Lemma. *Let A be a unital Banach algebra and let $a \in A$. The spectrum $\sigma(a)$ of a is a closed subset of the disc in the plane of centre the origin and radius $\|a\|$, and the map*

$$\mathbf{C} \setminus \sigma(a) \to A, \ \lambda \mapsto (a - \lambda)^{-1},$$

is differentiable.

Proof. If $|\lambda| > \|a\|$, then $\|\lambda^{-1}a\| < 1$, so $1 - \lambda^{-1}a$ is invertible, and therefore, so is $\lambda - a$. Hence, $\lambda \notin \sigma(a)$. Thus, $\lambda \in \sigma(a) \Rightarrow |\lambda| \leq \|a\|$. The set $\sigma(a)$ is closed, that is, $\mathbf{C} \setminus \sigma(a)$ is open, because $\mathrm{Inv}(A)$ is open in A. Differentiability of the map $\lambda \mapsto (a - \lambda)^{-1}$ follows from Theorem 1.2.3. □

The following result can be thought of as the fundamental theorem of Banach algebras.

1.2.5. Theorem (Gelfand). *If a is an element of a unital Banach algebra A, then the spectrum $\sigma(a)$ of a is non-empty.*

Proof. Suppose that $\sigma(a) = \emptyset$ and we shall obtain a contradiction. If $|\lambda| > 2\|a\|$, then $\|\lambda^{-1}a\| < \frac{1}{2}$, and therefore, $1 - \|\lambda^{-1}a\| > \frac{1}{2}$. Hence,

$$\|(1-\lambda^{-1}a)^{-1} - 1\| = \|\sum_{n=1}^{\infty}(\lambda^{-1}a)^n\|$$
$$\leq \frac{\|\lambda^{-1}a\|}{1-\|\lambda^{-1}a\|} \leq 2\|\lambda^{-1}a\| < 1.$$

Consequently, $\|(1-\lambda^{-1}a)^{-1}\| < 2$, and therefore,

$$\|(a-\lambda)^{-1}\| = \|\lambda^{-1}(1-\lambda^{-1}a)^{-1}\| < 2/|\lambda| < \|a\|^{-1}$$

($a \neq 0$ since $\sigma(a) = \emptyset$). Moreover, since the map $\lambda \mapsto (a-\lambda)^{-1}$ is continuous, it is bounded on the (compact) disc $2\|a\|\mathbf{D}$. Thus, we have shown that this map is bounded on all of $\mathbf{C}$; that is, there is a positive number M such that $\|(a-\lambda)^{-1}\| \leq M$ $(\lambda \in \mathbf{C})$.

If $\tau \in A^*$, the function $\lambda \mapsto \tau((a-\lambda)^{-1})$ is entire, and bounded by $M\|\tau\|$, so by Liouville's theorem in complex analysis, it is constant. In particular, $\tau(a^{-1}) = \tau((a-1)^{-1})$. Because this is true for all $\tau \in A^*$, we have $a^{-1} = (a-1)^{-1}$, so $a = a - 1$, which is a contradiction. □

It is easy to see that there are algebras in which not all elements have non-empty spectrum. For example, if $\mathbf{C}(z)$ denotes the field of quotients of $\mathbf{C}[z]$, then $\mathbf{C}(z)$ is an algebra, and the spectrum of z in this algebra is empty.

1.2.6. Theorem (Gelfand–Mazur). *If A is a unital Banach algebra in which every non-zero element is invertible, then $A = \mathbf{C}1$.*

Proof. This is immediate from Theorem 1.2.5. □

If a is an element of a unital Banach algebra A, its ***spectral radius*** is defined to be

$$r(a) = \sup_{\lambda \in \sigma(a)} |\lambda|.$$

By Remark 1.2.1, $r(ab) = r(ba)$ for all $a, b \in A$.

1.2.4. *Example.* If $A = C(\Omega)$, where Ω is a compact Hausdorff space, then $r(f) = \|f\|_\infty$ $(f \in A)$.

1.2.5. *Example.* Let $A = M_2(\mathbf{C})$ and

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $\|a\| = 1$, but $r(a) = 0$, since $a^2 = 0$.

1.2.7. Theorem (Beurling). *If a is an element of a unital Banach algebra A, then*

$$r(a) = \inf_{n\geq 1} \|a^n\|^{1/n} = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

Proof. If $\lambda \in \sigma(a)$, then $\lambda^n \in \sigma(a^n)$, so $|\lambda^n| \leq \|a^n\|$, and therefore, $r(a) \leq \inf_{n\geq 1} \|a^n\|^{1/n} \leq \liminf_{n\to\infty} \|a^n\|^{1/n}$.

Let Δ be the open disc in $\mathbf{C}$ centered at 0 and of radius $1/r(a)$ (we use the usual convention that $1/0 = +\infty$). If $\lambda \in \Delta$, then $1 - \lambda a \in \operatorname{Inv}(A)$. If $\tau \in A^*$, then the map

$$f\colon \Delta \to \mathbf{C},\ \lambda \mapsto \tau((1-\lambda a)^{-1}),$$

is analytic, so there are unique complex numbers λ_n such that

$$f(\lambda) = \sum_{n=0}^{\infty} \lambda_n \lambda^n \quad (\lambda \in \Delta).$$

However, if $|\lambda| < 1/\|a\| (\leq 1/r(a))$, then $\|\lambda a\| < 1$, so

$$(1-\lambda a)^{-1} = \sum_{n=0}^{\infty} \lambda^n a^n,$$

and therefore,

$$f(\lambda) = \sum_{n=0}^{\infty} \lambda^n \tau(a^n).$$

It follows that $\lambda_n = \tau(a^n)$ for all $n \geq 0$. Hence, the sequence $(\tau(a^n)\lambda^n)$ converges to 0 for each $\lambda \in \Delta$, and therefore *a fortiori*, it is bounded. Since this is true for each $\tau \in A^*$, it follows from the principle of uniform boundedness that $(\lambda^n a^n)$ is a bounded sequence. Hence, there is a positive number M (depending on λ, of course) such that $\|\lambda^n a^n\| \leq M$ for all $n \geq 0$, and therefore, $\|a^n\|^{1/n} \leq M^{1/n}/|\lambda|$ (if $\lambda \neq 0$). Consequently, $\limsup_{n\to\infty} \|a^n\|^{1/n} \leq 1/|\lambda|$. We have thus shown that if $r(a) < |\lambda^{-1}|$, then $\limsup_{n\to\infty} \|a^n\|^{1/n} \leq |\lambda^{-1}|$. It follows that $\limsup_{n\to\infty} \|a^n\|^{1/n} \leq r(a)$, and since $r(a) \leq \liminf_{n\to\infty} \|a^n\|^{1/n}$, therefore $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$. □

1.2.6. *Example.* Let A be the set of C^1-functions on the interval $[0,1]$. This is an algebra when endowed with the pointwise-defined operations, and a submultiplicative norm on A is given by

$$\|f\| = \|f\|_\infty + \|f'\|_\infty \qquad (f \in A).$$

It is elementary that A is complete under this norm, and therefore, A is a Banach algebra. Let $x\colon [0,1] \to \mathbf{C}$ be the inclusion, so $x \in A$. Clearly, $\|x^n\| = 1 + n$ for all n, so $r(x) = \lim(1+n)^{1/n} = 1 < 2 = \|x\|$.

Recall that if K is a non-empty compact set in $\mathbf{C}$, its complement $\mathbf{C} \setminus K$ admits exactly one unbounded component, and that the bounded components of $\mathbf{C} \setminus K$ are called the ***holes*** of K.

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