Lecture Notes in Control and Information Sciences

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ISBN 3-540-15595-3 Springer-Verlag Berlin Heidelberg New York Tokyo
ISBN 0-387-15595-3 Springer-Verlag New York Heidelberg Berlin Tokyo

Library of Congress Cataloging in Publication Data
Isidori, Alberto:
Nonlinear control systems: an introduction/A. Isidori.
Berlin; Heidelberg; New York; Tokyo: Springer 1985.
(Lecture notes in control and information sciences; Vol. 72)
NE: GT

Offsetprinting: Mercedes-Druck, Berlin
Binding: Lüderitz und Bauer, Berlin
2161/3020-543210

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CHAPTER I

LOCAL DECOMPOSITIONS OF CONTROL SYSTEMS

1. Introduction

In this section we review some basic results from the theory of linear systems, with the purpose of describing some fundamental properties which find close analogues in the theory of nonlinear systems.

Usually, a linear control system is described by equations of the form

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

in which the state x belongs to X, an n-dimensional vector space and the input u and the output y belong respectively to an m-dimensional vector space U and ℓ-dimensional vector space Y. The mappings $A : X \to X$, $B : U \to X$, $C : X \to Y$ are linear mappings.

Suppose that there exists a d-dimensional subspace V of X with the following property:

(i) V is invariant under the mapping A, i.e. is such that $Ax \in V$ for all $x \in V$;

then, it is known from linear algebra that there exists a basis for X (namely, any basis $(v_1,\ldots,v_n)$ with the property that $(v_1,\ldots,v_d)$ is also a basis for V) in which A is represented by means of a block-triangular matrix

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

whose elements on the lower (n-d) rows and left d columns are vanishing.

Moreover, if this subspace V is such that:

(ii) V contains the image of the mapping B, i.e. is such that $Bu \in V$ for all $u \in U$;

then, choosing again the same basis as before for X, regardless of the choice of basis in U, the mapping B is represented by a matrix

$$\begin{pmatrix} B_1 \\ 0 \end{pmatrix}$$

whose last n-d rows are vanishing.

Thus, if there exists a subspace V which satisfies (i) and (ii), then there exists a choice of coordinates for X in which the control system is described by a set of differential equations of the form

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1u$$

$$\dot{x}_2 = A_{22}x_2$$

By x_1 and x_2 we denote the d-vector and, respectively, the n-d vector formed by taking the first d and, respectively, the last n-d coordinates of a point x of X in the selected basis.

The representation thus obtained is particularly interesting when studying the behavior of the system under the action of the control u. At time T, the coordinates of x(T) are

$$x_1(T) = \exp(A_{11}T)x_1(0) + \int_0^T \exp(A_{11}(T-\tau))A_{12}\exp(A_{22}\tau)d\tau x_2(0) + \\ + \int_0^T \exp(A_{11}(T-\tau))B_1u(\tau)d\tau$$

$$x_2(T) = \exp(A_{22}T)x_2(0)$$

From this we see that the set of coordinates denoted with x_2 does not depend on the input u but only on the time T. The set of points that can be reached at time T, starting from x(0), under the action of the input lies inside the set of points of X whose x_2 coordinate is equal to $\exp(A_{22}T)x_2(0)$. In other words, if we let $x^o(T)$ denote the point of X reached at time T when $u(t) = 0$ for all $t \in [0,T]$, we observe that the state x(T) may be expressed as

$$x(T) = x^o(T) + v$$

where v is a vector in V. Therefore, the set of points that can be reached at time T, starting from x(0), lies inside the set

$$S_T = x^o(T) + V$$

Let us now make the additional assumption that the subspace V, which is the starting point of our considerations, is such that:

(iii) V is the smallest subspace which satisfies (i) and (ii) (i.e. is contained in any other subspace of X which satisfies both (i) and (ii)).

It is known from the linear theory that this happens if and only if

$$V = \sum_{i=0}^{n-1} \mathrm{Im}(A^i B)$$

and, moreover, that in this case the pair (A_{11},B_1) is a reachable pair, i.e. satisfies the condition

$$\mathrm{rank}(B_1 \; A_{11}B_1 \; \dots \; A_{11}^{d-1}B_1) = d$$

or, in other words, for each $x_1 \in \mathbb{R}^d$ there exists an input u, defined on $[0,T]$, such that

$$x_1 = \int_0^T \exp(A_{11}(T-\tau))B_1 u(\tau)d\tau$$

Then, if V is such that the condition (iii) is also satisfied, starting from x(0) we can reach at time T any state of the form $x^o(T) + v$ with $v \in V$ or, in other words, any state belonging to the set S_T. This set is therefore exactly the set of the states reachable at time T starting from x(0).

This result suggests the following considerations. Given a linear control system, let V be the smallest subspace of X satisfying (i) and (ii). Associated with V there is a *partition* of X into subsets of the form

$$x + V$$

with the property that each one of these subsets coincides with the set of points reachable at some time T starting from a suitable point of X. Moreover, these subsets have the structure of a d-dimensional *flat submanifold* of X.

An analysis similar to the one developed so far can be carried out by examining the interaction between state and output. In this case we consider a d-dimensional subspace W of X such that

(i) W is invariant under the mapping A
(ii) W is contained into the kernel of the mapping C (i.e. is such that $Cx = 0$ for all $x \in W$)
(iii) W is the largest subspace which satisfies (i) and (ii) (i.e. contains any other subspace of X which satisfies both (i) and (ii)).

Then, there is a choice of coordinates for X in which the control system is described by equations of the form

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1u$$

$$\dot{x}_2 = A_{22}x_2 + B_2u$$

$$y = C_2x_2$$

From this we see that the set of coordinates denoted with x_1 has no influence on the output y. Thus any two initial states whose last n-d coordinates coincide produce two identical outputs under any input, i.e. are indistinguishable. Actually, any two states whose last n-d coordinates coincide are such that their difference is an element of W and, then, we may conclude that any two states belonging to a set of the form x+W are indistinguishable.

Moreover, we know that the condition (iii) is satisfied if and only if

$$W = \bigcap_{i=0}^{n-1} \ker(CA^i)$$

and, if this is the case, the pair (C_2,A_{22}) is observable, i.e. satisfies the condition

$$\operatorname{rank}(C_2' \quad A_{22}'C_2' \ldots (A_{22}')^{d-1}C_2') = d$$

or, in other words,

$$C_{22}\exp(A_{22}t)x_2 \equiv 0 \Rightarrow x_2 = 0$$

Then, if two initial states are such that their difference does not belong to W, they may be distinguished from each other by the output produced under zero input.

Again we may synthesize the above discussion with the following considerations. Given a linear control system, let W be the largest subspace of X satisfying (i) and (ii). Associated with W there is a *partition* of X into subsets of the form

$$x + W$$

with the property that each one of these subsets coincides with the set of points that are indistinguishable from a fixed point of X. Moreover, these subsets have the structure of a d-dimensional *flat submanifold* of X.

In the following sections of this chapter and in the following chapter we shall deduce similar decompositions for nonlinear control systems.

2. Distributions on a Manifold

The easiest way to introduce the notion of distribution Δ on a manifold N is to consider a mapping assigning to each point p of N a *subspace* $\Delta(p)$ of the tangent space T_pN to N at p. This is not a rigorous definition, in the sense that we have only defined the domain N of Δ without giving a precise characterization of its codomain. Deferring for a moment the need for a more rigorous definition, we proceed by adding some conditions of regularity. This is imposed by assuming that for each point p of N there exist a neighborhood U of p and a set of smooth vector fields defined on U, denoted $\{\tau_i : i \in I\}$, with the property that,

$$\Delta(q) = \text{span}\{\tau_i(q) : i \in I\}$$

for all $q \in U$. Such an object will be called a smooth *distribution* on N. Unless otherwise noted, in the following sections we will use the term "distribution" to mean a smooth distribution.

Pointwise, a distribution is a linear object. Based on this property, it is possible to extend a number of elementary concepts related to the notion of subspace. Thus, if $\{\tau_i : i \in I\}$ is a set of vector fields defined on N, their *span*, written $\text{sp}\{\tau_i : i \in I\}$, is the distribution defined by the rule(*)

$$\text{sp}\{\tau_i : i \in I\} : p \mapsto \text{span}\{\tau_i(p) : i \in I\}$$

If Δ_1 and Δ_2 are two distributions, their *sum* $\Delta_1 + \Delta_2$ is defined by taking

$$\Delta_1 + \Delta_2 : p \mapsto \Delta_1(p) + \Delta_2(p)$$

and their *intersection* $\Delta_1 \cap \Delta_2$ by taking

$$\Delta_1 \cap \Delta_2 : p \mapsto \Delta_1(p) \cap \Delta_2(p)$$

(*) In order to avoid confusions, we use the symbol span{•} to denote any $\mathbb{R}$-linear combination of elements of some $\mathbb{R}$-vector space (in particular, tangent vectors *at a point*). The symbol sp{•} is used to denote a *distribution* (or a *codistribution*, see later).

A distribution Δ_1 *is contained* in the distribution Δ_2 and is written $\Delta_1 \subset \Delta_2$ if $\Delta_1(p) \subset \Delta_2(p)$ for all $p \in N$. A vector field τ *belongs* to a distribution Δ and is written $\tau \in \Delta$ if $\tau(p) \in \Delta(p)$ for all $p \in N$.

The *dimension* of a distribution Δ at $p \in N$ is the dimension of the subspace $\Delta(p)$ of T_pN.

Note that the span of a given set of smooth vector fields is a smooth distribution. Likewise, the sum of two smooth distributions is smooth. However, the intersection of two such distributions may fail to be smooth. This may be seen in the following example.

(2.1) *Example.* Let $M = \mathbb{R}^2$, and

$$\Delta_1 = sp\{\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\}$$

$$\Delta_2 = sp\{(1+x_1)\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\}$$

Then we have

$$(\Delta_1 \cap \Delta_2)(x) = \{0\} \quad \text{if} \quad x_1 \neq 0$$

$$(\Delta_1 \cap \Delta_2)(x) = \Delta_1(x) = \Delta_2(x) \quad \text{if} \quad x_1 = 0$$

This distribution is not smooth because it is not possible to find a smooth vector field on $\mathbb{R}^2$ which is zero everywhere but on the line $x_1 = 0$. □

Since sometimes it is useful to take the intersection of smooth distributions Δ_1 and Δ_2 , one may overcome the problem that $\Delta_1 \cap \Delta_2$ is possibly non-smooth with the aid of the following concepts. Suppose Δ is a mapping which assigns to each point $p \in N$ a subspace $\Delta(p)$ of T_pN and let $M(\Delta)$ be the set of all smooth vector fields defined on N which at p take values in $\Delta(p)$, i.e.

$$M(\Delta) = \{\tau \in V(N):\tau(p) \in \Delta(p) \text{ for all } p \in N\}$$

Then, it is not difficult to see that the span of $M(\Delta)$, in the sense defined before, is a smooth distribution contained in Δ.

(2.2) *Remark.* Recall that the set $V(N)$ of all smooth vector fields defined on N may be given the structure of a vector space over $\mathbb{R}$ and, also, the structure of a module over $C^\infty(N)$, the ring of all smooth real-valued functions defined on N. The set $M(\Delta)$ defined before (which is non-empty because the zero element of $V(N)$ belongs to $M(\Delta)$ for any

Δ) is a subspace of the vector space V(N) and a submodule of the module V(N). From this is it easily seen that the span of $M(\Delta)$ is contained in Δ. □

Note that if Δ' is any smooth distribution contained in Δ, then Δ' is contained in the span of $M(\Delta)$, so the span of $M(\Delta)$ is actually the *largest* smooth distribution contained in Δ. To identify this distribution we shall henceforth use the notation

$$\mathrm{smt}(\Delta) \stackrel{\Delta}{=} \mathrm{sp}\, M(\Delta)$$

i.e. we look at the span of $M(\Delta)$ as the "smoothing" of Δ. Note also that if Δ is smooth, then $\mathrm{smt}(\Delta) = \Delta$.

Thus, if $\Delta_1 \cap \Delta_2$ is non-smooth, we shall rather consider the distribution $\mathrm{smt}(\Delta_1 \cap \Delta_2)$.

(2.3) *Remark.* Note that $M(\Delta)$ may not be the unique subspace of V(N), or submodule of V(N), whose span coincides with $\mathrm{smt}(\Delta)$. But if M' is any other subspace of V(N), or submodule of V(N), with the property that $\mathrm{sp}\, M' = \mathrm{smt}(\Delta)$, then $M' \subset M(\Delta)$.

(2.4) *Example.* Let $N = \mathbb{R}$, and

$$\Delta = \mathrm{sp}\{x \frac{\partial}{\partial x}\}$$

Then $M(\Delta)$ is the set of all vector fields of the form $c(x)\frac{\partial}{\partial x}$ where $c(x)$ is a smooth function defined on $\mathbb{R}$ which vanishes at $x = 0$. Clearly Δ is smooth and coincides with $\mathrm{smt}(\Delta)$. There are many submodules of $V(\mathbb{R})$ which span Δ, for instance

$$M_1 = \{\tau \in V(\mathbb{R}) : \tau(x) = c(x)x \frac{\partial}{\partial x} \text{ and } c \in C^\infty(\mathbb{R})\}$$

$$M_2 = \{\tau \in V(\mathbb{R}) : \tau(x) = c(x)x^2 \frac{\partial}{\partial x} \text{ and } c \in C^\infty(\mathbb{R})\}$$

Both are submodules of $M(\Delta)$, M_2 is a submodule of M_1 but M_1 is not a submodule of M_2 because is not possible to express every function $c(x)x$ as $\hat{c}(x)x^2$ with $\hat{c} \in C^\infty(\mathbb{R})$. □

(2.5) *Remark.* The previous considerations enable us to give a rigorous definition of a smooth distribution in the following way. A smooth distribution is a submodule M of V(N) with the following property: if θ is a smooth vector field such that for all $p \in N$

$$\theta(p) \in \mathrm{span}\{\tau(p) : \tau \in M\}$$

then θ belongs to M. □

Other important concepts associated with the notion of distribution are the ones related to the "behavior" of a given Δ as a "function" of p. We have already seen how it is possible to characterize the quality of being smooth, but there are other properties to be considered.

A distribution Δ is *nonsingular* if there exists an integer d such that

$$\dim \Delta(p) = d \tag{2.6}$$

for all $p \in N$. A singular distribution, i.e. a distribution for which the above condition is not satisfied, is sometimes called a distribution of variable dimension. If a distribution Δ is such that the condition (2.6) is satisfied for all p belonging to an open subset U of N, then we say that Δ is nonsingular on U. A point p is a *regular* point of a distribution Δ if there exists a neighborhood U of p with the property that Δ is nonsingular on U.

There are some interesting properties related to these notions, whose proof is left to the reader.

(2.7) *Lemma*. Let Δ be a smooth distribution and p a regular point of Δ. Suppose $\dim \Delta(p) = d$. Then there exist an open neighborhood U of p and a set $\{\tau_1,\ldots,\tau_d\}$ of smooth vector fields defined on U with the property that every smooth vector field τ belonging to Δ admits on U a representation of the form

$$\tau = \sum_{i=1}^{d} c_i \tau_i \tag{2.8}$$

where each c_i is a real-valued smooth function defined on U. □

A set of d vector fields which makes (2.8) satisfied will be called *a set of local generators* for Δ at p.

(2.9) *Lemma*. The set of all regular points of a distribution Δ is an open and dense submanifold of N.

(2.10) *Lemma*. Let Δ_1 and Δ_2 be two smooth distributions with the property that Δ_2 is nonsingular and $\Delta_1(p) \subset \Delta_2(p)$ at each point p of a dense submanifold of N. Then $\Delta_1 \subset \Delta_2$.

(2.11) *Lemma*. Let Δ_1 and Δ_2 be two smooth distributions with the property that Δ_1 is nonsingular, $\Delta_1 \subset \Delta_2$ and $\Delta_1(p) = \Delta_2(p)$ at each point p of a dense submanifold of N. Then $\Delta_1 = \Delta_2$. □

We have seen before that the intersection of two smooth distributions may fail to be smooth. However, around a regular point this cannot happen, as we see from the following result.

(2.12) *Lemma.* Let p be a regular point of Δ_1 , Δ_2 and $\Delta_1 \cap \Delta_2$. Then there exists a neighborhood U of p with the property that $\Delta_1 \cap \Delta_2$ restricted to U is smooth. □

A distribution is *involutive* if the Lie bracket $[\tau_1,\tau_2]$ of any pair of vector fields τ_1 and τ_2 belonging to Δ is a vector field which belongs to Δ, i.e. if

$$\tau_1 \in \Delta,\ \tau_2 \in \Delta \Rightarrow [\tau_1,\tau_2] \in \Delta$$

(2.13) *Remark.* It is easy to see that a nonsingular distribution of dimension d is involutive if and only if, at each point p, any set of local generators $\tau_1,\dots,\tau_d$ defined on a neighborhood U of p is such that

$$[\tau_i,\tau_j] = \sum_{\ell=1}^{d} c_{ij}^{\ell}\tau_\ell$$

where each c_{ij}^{ℓ} is a real-valued smooth function defined on U. □

If f is a vector field and Δ a distribution on N we denote by $[f,\Delta]$ the distribution

$$[f,\Delta] = \mathrm{sp}\{[f,\tau] \in V(N) : \tau \in \Delta\} \tag{2.14}$$

Note that $[f,\Delta]$ is a smooth distribution, even if Δ is not. Using this notation, one can say that a distribution is involutive if and only if $[f,\Delta] \subset \Delta$ for all $f \in \Delta$.

Sometimes, it is useful to work with objects that are dual to the ones defined above. In the same spirit of the definition given at the beginning of this section, we say that a *codistribution* Ω on N is a mapping assigning to each point p of N a subspace $\Omega(p)$ of the cotangent space $T_p^*(N)$. A smooth codistribution is a codistribution Ω on N with the property that for each point p of N there exist a neighborhood U of p and a set of smooth covector fields (smooth one-forms) defined on U, denoted $\{\omega_i : i \in I\}$, such that

$$\Omega(q) = \mathrm{span}\{\omega_i(q) : i \in I\}$$

for all $q \in U$.

In the same manner as we did for distributions we may define the dimension of a codistribution at p, and construct codistributions by taking the span of a given set of covector fields, or else by adding or intersecting two given codistributions, etc. always looking at a pointwise characterization of the objects we are dealing with.

Sometimes, one can construct codistributions starting from given distributions and conversely. The natural way to do this is the following: given a distribution Δ on N, the *annihilator* of Δ, denoted $\Delta^{\perp}$, is the codistribution on N defined by the rule

$$\Delta^{\perp} : p \mapsto \{ v^* \in T_p^*N : \langle v^*, v \rangle = 0 \text{ for all } v \in \Delta(p) \}$$

Conversely, the *annihilator* of Ω, denoted $\Omega^{\perp}$, is the distribution defined by the rule

$$\Omega^{\perp} : p \mapsto \{ v \in T_pN : \langle v^*, v \rangle = 0 \text{ for all } v^* \in \Omega(p) \}$$

Distributions and codistributions thus related possess a number of interesting properties. In particular, the sum of the dimensions of Δ and of $\Delta^{\perp}$ is equal to the dimension of N. The inclusion $\Delta_1 \subset \Delta_2$ is satisfied if and only if the inclusion $\Delta_1^{\perp} \supset \Delta_2^{\perp}$ is satisfied. The annihilator $(\Delta_1 \cap \Delta_2)^{\perp}$ of an intersection of distributions is equal to the sum $\Delta_1^{\perp} + \Delta_2^{\perp}$.

Like in the case of the distributions, some care is required when dealing with the quality of being smooth for codistributions constructed in some of the ways we described before. Thus it is easily seen that the span of a given set of smooth covector fields, as well as sum of two smooth codistributions is again smooth. But the intersection of two such codistributions may not need to be smooth.

Moreover, the annihilator of a smooth distribution may fail to be smooth, as it is shown in the following example.

(2.15) *Example.* Let $N = \mathbb{R}$

$$\Delta = \text{sp}\{x \frac{\partial}{\partial x}\}$$

Then

$$\Delta^{\perp}(x) = \{0\} \quad \text{if } x \neq 0$$

$$\Delta^{\perp}(x) = T_x^*N \quad \text{if } x = 0$$

and we see that $\Delta^{\perp}$ is not smooth because it is not possible to find a smooth covector field on $\mathbb{R}$ which is zero everywhere but on the point

$x = 0$. □

Or, else, the annihilator of a smooth codistribution may not be smooth, as in the following example.

(2.16) *Example.* Consider again the two distributions Δ_1 and Δ_2 described in the Example (2.1). One may easily check that

$$\Delta_1^{\perp} = \mathrm{sp}\{dx_1 - dx_2\}$$

$$\Delta_2^{\perp} = \mathrm{sp}\{dx_1 - (1+x_1)dx_2\}$$

The intersection $\Delta_1 \cap \Delta_2$ is not smooth but its annihilator $\Delta_1^{\perp} + \Delta_2^{\perp}$ is smooth, because both $\Delta_1^{\perp}$ and $\Delta_2^{\perp}$ are smooth. □

One may easily extend Lemmas (2.7) to (2.12). In particular, if p is a regular point of a codistribution Ω and $\dim \Omega(p) = d$, then it is possible to find an open neighborhood U of p and a set $\{\omega_1,\ldots,\omega_d\}$ of smooth covector fields defined on U, such that every smooth covector field ω belonging to Ω can be expressed on U as

$$\omega = \sum_{i=1}^{d} c_i \omega_i$$

where each c_i is a real-valued smooth function defined on U. The set $\{\omega_1,\ldots,\omega_d\}$ is called a set of local generators for Ω at p.

We have seen before that the annihilator of a smooth distribution Δ may fail to be smooth. However, around a regular point of Δ this cannot happen, as we see from the following result.

(2.17) *Lemma.* Let p be a regular point of Δ. Then p is a regular point of $\Delta^{\perp}$ and there exists a neighborhood U of p with the property that $\Delta^{\perp}$ restricted to U is smooth. □

We conclude this section with some notations that are frequently used. If f is a vector field and Ω a codistribution on N we denote by $L_f\Omega$ the smooth codistribution

$$L_f\Omega = \mathrm{sp}\{L_f\omega \in V^*(N) : \omega \in \Omega\} \tag{2.18}$$

If h is a real-valued smooth function defined on N, one may associate with h a distribution, written $\ker(h_*)$, defined by

$$\ker(h_*) : p \mapsto \{v \in T_pN : h_*v = 0\}$$

One may also associate with h a codistribution, taking the span of the covector field dh. It is easy to verify that the two objects thus defined are one the annihilator of the other, i.e. that

$$(sp(dh))^{\perp} = \ker(h_*).$$

3. Frobenius Theorem

In this section we shall establish a correspondence between the notion of distribution on a manifold N and the existence of partitions of N into lower dimensional submanifolds. As we have seen at the beginning of this chapter, partitions of the state space into lower dimensional submanifolds are often encountered when dealing with reachability and/or observability of control systems.

We begin our analysis with the following definition. A *nonsingular* d-dimensional distribution Δ on N is *completely integrable* if at each $p \in N$ there exists a cubic coordinate chart (V,ξ) with coordinate functions $\xi_1,\ldots,\xi_n$, such that

$$(3.1) \qquad \Delta(q) = \text{span}\{(\frac{\partial}{\partial\xi_1})_q,\ldots,(\frac{\partial}{\partial\xi_d})_q\}$$

for all $q \in V$.

There are two important consequences related to the notion of completely integrable distribution. First of all, observe that if there exists a cubic coordinate chart (V,ξ), with coordinate functions $\xi_1,\ldots,\xi_n$, such that (3.1) is satisfied, then any *slice* of V passing through any point p of V and defined by

$$(3.2) \qquad S_p = \{q \in V: \xi_i(q) = \xi_i(p);\ i = d+1,\ldots,n\}$$

(which is a d-dimensional imbedded submanifold of N), has a tangent space which, at any point q, coincides with the subspace $\Delta(q)$ of T_qN.

Since the set of all such slices is a *partition* of V, we may see that a completely integrable distribution Δ induces, locally around each point $p \in N$, a partition into lower dimensional submanifolds, and each of these submanifolds is such that its tangent space, at each point, agrees with the distribution Δ at that point.

The second consequence is that a completely integrable distribution is *involutive*. In order to see this we use the definition of involutivity and compute the Lie bracket of any pair of vector fields belonging to Δ. For, recall that in the ξ coordinates, any vector field τ defined on N is represented by a vector of the form