NONLINEARITY AND FUNCTIONAL ANALYSIS

Lectures on Nonlinear Problems in Mathematical Analysis

Melvin S. Berger

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Belfer Graduate School
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New York, New York

ACADEMIC PRESS New York San Francisco London 1977

A Subsidiary of Harcourt Brace Jovanovich, Publishers

ACADEMIC PRESS, INC.
111 Fifth Avenue, New York, New York 10003

United Kingdom Edition published by
ACADEMIC PRESS, INC. (LONDON) LTD.
24/28 Oval Road. London NW1

Library of Congress Cataloging in Publication Data

Berger, Melvyn, Date
Nonlinearity and functional analysis.

(Pure and applied mathematics)
Bibliography: p.
Includes index.
1. Mathematical analysis. 2. Nonlinear theories. I. Title.
QA300.B458 515'.7 76-26039
ISBN 0-12-090350-4

PRINTED IN THE UNITED STATES OF AMERICA

PREFACE

For many decades great mathematical interest has focused on problems associated with linear operators and the extension of the well-known results of linear algebra to an infinite-dimensional context. This interest has been crowned with deep insights, and the substantial theory that has been developed has had a profound influence throughout the mathematical sciences. However when one drops the assumption of linearity, the associated operator theory and the many concrete problems associated with such a theory represent a frontier of mathematical research. Nonetheless, the fundamental results so far obtained in this direction already form a deep and beautiful extension of this linear theory. Just as in the linear case, these results were inspired by, and are highly relevant to concrete problems in mathematical analysis. The object of the lectures represented here is a systematic description of these fundamental nonlinear results and their applicability to a variety of concrete problems taken from various fields of mathematical analysis.

Here I use the term "mathematical analysis" in the broadest possible sense. This usage is in accord with the ideas of Henri Poincaré (one of the great pioneers of our subject). Indeed, by carefully scrutinizing the specific nonlinear problems that arise naturally in the study of the differential geometry of real and complex manifolds, classical and modern mathematical physics, and the calculus of variations, one is able to discern recurring patterns that inevitably lead to deep mathematical results.

From an abstract point of view there are basically two approaches to the subject at hand. The first, as mentioned above, consists of extending specific results of linear functional analysis associated with the names of Fredholm, Hilbert, Riesz, Banach, and von Neumann to a more general nonlinear context. The second approach consists of viewing the subject matter as an infinite-dimensional version of the differential geometry of manifolds and mappings between them. Obviously, these approaches are closely related, and when used in conjunction with modern topology, they form a mode of mathematical thought of great power.

Finally, over and above these two approaches, there are phenomena that are genuinely both nonlinear and infinite dimensional in character. A framework for understanding such facts is still evolving.

The material to be described is divided into three parts, with each part containing two chapters. Part I is concerned first with the motivation and preliminary mathematical material necessary to understand the context of later developments in the book and second with providing a rudimentary calculus and classification of nonlinear operators. Part II deals with local analysis. In Chapter 3, I treat various infinite-dimensional extensions of the classical inverse and implicit function theorems as well as Newton's, the steepest descent and majorant methods for the study of operator equations. In Chapter 4, I turn attention to those parameter-dependent perturbation phenomena related to bifurcation and singular perturbation problems. In this chapter the use of topological ("transcendental") methods makes its first decisive appearance. The third and final part of the book describes analysis in the large and shows the necessity of combining concrete analysis with transcendental methods. Chapter 5 develops global methods that are applicable to general classes of operators. In particular, it treats the various theories and applications of the mapping degree and its recent extensions involving higher homotopy groups of spheres as well as linearization and projection methods. Chapter 6 describes the calculus of variations in the large and its current developments in modern critical point theory. This material evolves naturally from minimization and isoperimetric problems involving critical points of higher type.

A main object of the text is the application of the abstract results obtained to resolve interesting problems of geometry and physics. The applications represented have been chosen with regard both to their intrinsic interest and to their relation to the abstract material presented in the text. In many cases the specific examples require an extension of theory and so serve as a motive force for further developments. It is hoped that the deeper and more complicated applications included will enhance the value and interest of this rapidly developing subject.

Moreover, I have chosen a few nonlinear problems as models for our abstract developments. These include

(i) the determination of periodic solutions for systems of nonlinear ordinary differential equations,

(ii) Dirichlet's problem for various semilinear elliptic partial differential equations,

(iii) the differential geometric problem of determining the "simplest" metric on a given compact manifold (simplest meaning constant curvature here),

(iv) the structure of the solutions of von Kármán's equations of nonlinear elasticity.

All these models illustrate the need for new theoretical developments and more subtle and incisive methods of study. In addition, the classical nature of these problems indicates the tremendous scope for research on the abstract essentials of less classical nonlinear problems.

Many of the abstract results and applications described in the text are of recent origin, but I hope nonetheless that they form a unified pattern of development that differs from existing monographs on the subject. The choice of subject matter presented here has been highly subjective, and, in order to keep the text to a reasonable number of pages, many important topics have been treated only superficially, if at all. Thus material dealing with ordered Banach spaces, variational inequalities, convex analysis, monotone mappings, and parabolic and hyperbolic partial differential equations has been largely avoided. Moreover, these topics have been well covered by a number of recent monographs and survey articles. In a somewhat different vein, I have avoided applications too special to illustrate the general principles addressed. An example is two-point boundary value problems for a single second-order nonlinear differential equation. Such problems can (for example) be successfully treated by phase-plane methods. Finally, the recent "Euclidean" field theory methods of modern physics have shown that nonlinear hyperbolic systems can often be treated in terms of the nonlinear elliptic boundary value problems treated here.

This book has been written over a period of years, so that various kinds of misprints inevitably arise. I ask the reader to inform me of any such misprint, so that an errata list may be prepared. Yet, I hope the material described here has sufficient coherence, intrinsic interest, and attractiveness to provide the reader with a framework for further excursions into nonlinear analysis.

Many interesting nonlinear problems and illustrative examples have had to be deleted from the text to keep the book within manageable size. I hope in the near future to complete another volume containing these items as well as instructive but more routine problems. This volume will also contain a more complete bibliography.

Finally, I would like to thank all of those who helped in producing this book. They include D. Westreich, R. Plastock, E. Podolak, J. vande Koppel, T. Goldring, S. Kleiman, A. Steif, S. Nachtigall, M. Schechter, L. E. Fraenkel, S. Karlin, W. B. Gordon, A. Wightman, and last but by no means least, my editors at Academic Press. This book could not have been written without the generous financial support of the Air Force Office of Scientific Research and the National Science Foundation. To both organizations I extend my hearty thanks.

NOTATION AND TERMINOLOGY

Ω	an open subset of real N-dimensional Euclidean space $\mathbb{R}^N$
$\mathfrak{M}^N$	an N-dimensional smooth manifold
$x = (x_1, \ldots x_N)$	the Cartesian coordinates of a point in $\mathbb{R}^N$
$D_j = \partial / \partial x_j$	the elementary partial derivative operators acting on functions defined on Ω
$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$	multi-index
$\|\alpha\|$	$\Sigma_{i=1}^N \alpha_i$
D^α	$\Pi_{i=1}^N D_i^{\alpha_i}$
$F(x, D^\beta u, D^m u)$	a differential operator of order m depending explicitly on the elementary higher order differential operators D^α of order m with $\|\beta\| < m$
X	some linear vector space of functions
linear operator F	$F(\alpha f + \beta g) = \alpha F(f) + \beta F(g)$ for each f, $g \in X$, and scalars α, β
nonlinear operator F	an operator F that is not necessarily linear
quasilinear differential operator $F(x, D^\beta f, D^\alpha f)$	F is a linear differential operator when regarded as a function of the elementary differential operators $D^\alpha f$ of order m alone
differential equation defined on Ω	an equation between two differential operators which must hold at each point of Ω
classical solution	a (sufficiently smooth) function (defined on Ω), which satisfies the equation at each point of Ω
$\|x\|$	length of a vector $x \in \mathbb{R}^N$
$\|\|u\|\|$	the norm of an element u of a Banach space X
absolute constant	used in connection with the inequality $F(x) \leq cG(x)$ to mean that the c does not depend on x as x varies
seminorm	a nonnegative function g defined on X such that $g(\alpha x) = \|\alpha\| g(x)$ and $g(x+y) \leq g(x) + g(y)$

SUGGESTIONS FOR THE READER

The present book is intended as a synthesis between certain aspects of mathematical analysis and other areas of science. Such a synthesis requires much motivation and a creative approach usually not found in textbooks.

Thus those parts of Chapters 1 and 2 that provide background material and preliminary information need not be read straight through. Rather the reader is encouraged to skip around to find bits of knowledge that excite his interest and to pursue these directly into the later Chapters. When necessary, the reader should return to Part I to pick up necessary information. Reading this book is not intended to be a linear experience!

Chapter 3 is intended to be abstract in nature and to help develop a facility in utilizing the "functional analysis" language. The first three sections of Chapter 3 form a necessary prerequisite for all that follows. In contrast, Chapter 4 is applications-oriented throughout. Indeed, a proper understanding of parameter-dependent local analysis requires careful thinking about specific classic model problems. Again the reader can choose only those applications that fit his or her interest.

Part III can be read in separate pieces. Chapter 5, for example, contains three separate lines of development: Section 5.1, Section 5.2, and Sections 5.3–5.5 (a deep study requiring, of course, a blending of each strand). Similarly Chapter 6 divides naturally in three parts: Sections 6.1–6.2, Sections 6.3–6.4, and Sections 6.5–6.7. The first two parts do not make use of topological methods, but such methods are essential for the third.

The reader is expected to have some prerequisite knowledge of conventional linear functional analysis, ordinary and partial differential equations. Some acquaintance with undergraduate physics and differential geometry will be helpful in understanding the applications. These applications are treated rather tersely and with varying degrees of thoroughness. A comparison with more detailed and traditional treatments of each application will prove helpful. My idea has been to provide a sense of the scope, utility, and diversity of the subject matter without obscuring the key ideas.

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PART I

PRELIMINARIES

Many problems arising naturally in differential geometry and mathematical physics as well as in many other areas of science involve the study of solutions of systems of nonlinear differential equations. However, since most of these systems are "nonintegrable," in the sense that their solutions cannot be written in closed form, classical methods of studying such systems generally fail. Thus new methods of study are required. In recent years a new approach to these problems has proved both relatively successful and straightforward. The approach consists essentially in this: A given problem is reformulated in the language of function spaces; this abstracted problem is then analyzed as completely as possible by the methods of functional analysis, and the results obtained are then retranslated into statements concerning the original problem. The generality thus attained is important in several respects. First, a given problem is stripped of extraneous data, so that the analytic core of the problem is revealed. Secondly, seemingly diverse problems are shown to be specializations of the same theoretical ideas. Finally, the abstract structures that lie at the foundation of the study of novel nonlinear phenomena can be clearly ascertained. In the sequel, we shall describe this circle of ideas as well as the resulting interplay between (nonlinear) functional analysis and concrete problems.

The aim of Part I The subject matter to be discussed is distinguished from many other mathematical areas by the mixing of various "structures" inherent in even the simplest examples. Consequently, although most of the problems presented here are easily stated, the number of prerequisites necessary to the adequate understanding of the solution to a given problem may be quite large. Thus the aim of Part I is fourfold:

(i) to set out these prerequisites in a systematic way;

(ii) to motivate the various specific problems to be studied in the sequel;

(iii) to indicate the steps necessary to reformulate a specific problem in terms of appropriate abstract nonlinear operators;

(iv) to develop an elementary calculus for these abstract operators.

The first two points are treated in Chapter 1, while the last two are covered in Chapter 2.

CHAPTER 1

BACKGROUND MATERIAL

This chapter is divided into six sections. The first two sections list a number of classical geometric and physical nonlinear problems, as well as the typical difficulties encountered in studying such problems. Next we summarize the results from linear functional analysis that will be useful in the sequel. Then we review the regularity results for linear elliptic partial differential equations that have proven invaluable for the successful application of functional analysis to the nonlinear problems discussed in the first section. Finally we survey the basic facts concerning mappings between finite-dimensional spaces (and, in particular, results from topology) that will be needed in the text.

1.1 How Nonlinear Problems Arise

Before commencing a systematic study of nonlinearity, it is of interest to mention some important sources of the problems, some of which will be discussed in the sequel. Three classic sources of nonlinear problems are mentioned below: first, differential-geometric problems, in which nonlinearity enters naturally via curvature considerations; next, mathematical problems of classical and modern physics; and finally problems of the calculus of variations involving nonquadratic functionals. Of course these sources are not exhaustive, and the mathematical aspects of fields such as economics, genetics, and biology offer entirely new nonlinear phenomena (see the Notes at the end of this chapter).

1.1A Differential-geometric sources

Differential-geometric problems associated with the effects of curvature are a rich and historic source of nonlinear differential systems. The following examples indicate their scope:

(i) Geodesics on manifolds Consider the simple hypersurface S defined by setting $S = \{x \mid x \in \mathbb{R}^N, f(x) = 0\}$, where $f(x)$ is a C^2 real-valued function defined on $\mathbb{R}^N$ such that $|\nabla f| \neq 0$ on S. The geodesics on S are characterized as curves $g = x(t)$ on S that are critical points of the arc length functional. Geometrically, they are characterized by the property that the principal normal of g coincides with the normal of S. Analytically, geodesics are found as solutions of the following system of N ordinary differential equations:

$$\text{(1.1.1)} \qquad \text{(a)} \quad x_{tt} + \mu(t)\,\nabla f(x) = 0, \qquad \text{(b)} \quad f(x(t)) = 0,$$

where $\mu(t)$ is some real-valued function of t.

Apart from a few exceptional cases this system depends *nonlinearly* on $x(t)$. For example, let us determine the function $\mu(t)$ of (1.1.1) in terms of f. In fact, if we differentiate the relation $f(x(t)) = 0$ twice with respect to t, we find

$$H(f)x_t \cdot x_t + \nabla f \cdot x_{tt} = 0,$$

where $H(f)$ denotes the Hessian matrix $(\partial^2 f/\partial x_i\, \partial x_j)$ of f. Thus (1.1.1) implies that

$$\mu(t) = \{H(f)x_t \cdot x_t\}|\nabla f|^{-2}.$$

Consequently we find $|\nabla f|^2 \mu(t) = H(f)x_t \cdot x_t$. Thus the system (1.1.1) is nonlinear in x unless S is either a sphere so that $\mu(t)$ is a constant, or a hyperplane in which case $\mu(t) \equiv 0$. If S is an ellipsoid, Jacobi showed that the resulting system (1.1.1) could be explicitly solved in terms of elliptic functions. However, such integrable systems are rare; and the study of geodesics for hypersurfaces differing only slightly from an ellipsoid requires new and quite refined methods of study. More generally, if $(\mathfrak{M}^N, g)$ denotes an N-dimensional differentiable manifold with the Riemannian metric

$$ds^2 = \sum_{i,j=1}^{N} g_{ij}(x)\, dx_i\, dx_j,$$

then the geodesics on $(\mathfrak{M}^N, g)$ can be found as the solutions of the nonlinear system

$$\text{(1.1.2)} \qquad \ddot{x}_i + \sum_{j,k} \Gamma^i_{jk} \dot{x}_j \dot{x}_k = 0 \qquad (i, j, k = 1, 2, \ldots, N),$$

where Γ^i_{jk} denotes the so-called Christoffel symbol of the second kind. These symbols can be computed in terms of the functions g_{ij} and their derivatives. Thus the geodesics on $(\mathfrak{M}^N, g)$, defined by the equation (1.1.2), are *directly* related to the intrinsic metric and consequently to the curvature properties of $(\mathfrak{M}^N, g)$.

The study of the solutions of (1.1.2) in terms of the geometry and topology of $(\mathfrak{M}^N, g)$ has been a motive force in the discovery of many