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PREFACE

The last decade or so has been a very exciting time for those of us who have been engaged in researching applied numerical methods. It is within this time frame that the "boundary integral equation method" has risen from virtual obscurity, as a technique known and practised by just a few talented individuals in engineering and physics, to a position that places it as one of the predominant numerical methods in engineering research work. This author is confident that, at its present rate of growth in popularity, this now refined and updated method, most popularly known by the generic term "boundary element method" (or simply BEM), will in the next decade rival the finite element method as the most popular commercial numerical technique used by engineers. The author has also seen a major educational challenge arise as a result of this rapid growth.

Anytime there is a massive move forward in any endeavor, there are likely to be more passengers than there are vehicles. Unfortunately, the situation that exists in BEM is not an exception to this rule. Although there has been a mild proliferation of excellent texts and research collections written on BEM since 1975 (see the "Refer- and Supplemental Literature" section), there are still those who find the subject nearly impossible to break into initially. My experience, over the past eight years of being heavily involved in BEM research and over the past four years of trying to teach BEM, shows me that the mathematical content of the subject is the major reason for the existence of the educational barrier. I recall being asked to conduct a course on BEM this past year to a research group headed by an engineer (who is, by the way, also a part-time finite element teacher) holding a Ph.D. from one of the United States' most prestigious universities. His specific request of me was to help get him to the level where he could simply "read the books on the boundary elements." Another Ph.D. researcher in this same group wanted to "finally, once and for all, know what a Green's function is." I have heard similar remarks and complaints uttered in casual conversations about research topics with many of my (otherwise well-educated) colleagues in industry, academia, and in pure research.

The problem that exists, and the primary motivation for this author writing this monograph, is not truly the level of the mathematics, but the type of mathematics involved in BEM. Rizzo stated correctly some time ago (1975) that the math involved in BEM is not the type normally associated with getting answers to engineering problems; nor is the math beyond the abilities of graduate level engineers. The fact of the matter is that it is quite easy to neglect subjects such as Green's Lemma, contour integration, and especially Green's Functions in a typical program of engineering graduate study. It is even more typical to be exposed to these subjects, but never have to use them, and consequently, find oneself exceedingly uncomfortable when confronted with an equation having the unknown quantity residing under a definite integral! While the mathematical complexities can be avoided to some extent in the teaching of finite differences and finite elements, it is not easy to bypass them in boundary elements. Also, it is not particularly desirable to do so. Much of the richness of BEM is in its mathematical flavor, and the potential researcher in the area must make the sacrifice of learning and using a little more mathematics than that to which he may be accustomed.

Thus, as one way to address this problem, the first three to four chapters of this text is offered. This work represents the product of four years of effort in teaching boundary elements at the graduate level to a wide variety of engineering students, representing the Civil, Mechanical, Aerospace, Engineering Mechanics, Chemical, Metallurgical, and Agricultural Engineering disciplines. (One physicist has attended my lectures as well.) Also, as mentioned, the course has been taught successfully to special groups of research engineers already holding terminal degrees. The mode of teaching displayed by the notes is a casual, unpresumptuous, natural-train-of-thought approach to the subject. So that the newcomer can digest the material comfortably, the development proceeds slowly; but steadfastly and without sacrificing the mathematical quality needed for the reader to progress further in BEM. The deliberately unsophisticated style leads the reader through the evolution of boundary integral methods from a very basic level. Certain sections of the text, those prefaced by a "*", may in fact be too basic and contain peripheral subjects that the reader is likely to have seen elsewhere. These sections may be skipped over if this is the case. The text begins by looking at simple integral equations, how they come about, one-dimensional examples, and how to solve them. The text then progresses through a general discussion on the theory of approximation via the method of weighted residuals. The climax occurs in Chapter 3 when the rudimentary developments of the Laplace and Poisson equations are made. Some readers will surely find the text overly explanatory and the example problems worked out in the most agonizing, tedious detail. Be assured that this "overkill" is strictly intentional, and exists in order to make sure that no impediment to the self-teaching of the subject is left unconfronted. Homework problems are scattered liberally throughout the text and the reader is strongly advised to take the time to work these. If the reader has mastered Chapter 3, he should find the literature on BEM considerably easier to grasp. The coursework portion of the text stops with the development of the 2-D and axisymmetric orthotropic Poisson's equation with a consideration of piecewise nonhomogeneous media. Chapter 4 then illustrates the actual use of BEM for several problems governed by Laplace's equation. The premise that the author is using in curtailing the teaching portion of the text at this point is that this amount of development should be sufficient to guide the reader into the more complicated physical problems of engineering. At this stage, the reader who is still interested in BEM, but not necessarily in Poisson's equation, should look to the references for a text addressing the application he has in mind.

The remainder of the text is either for the person who is still interested in the Laplace-Poisson type equation or for the person generally interested in making BEM even simpler to use. Chapter 5 represents the secondary motive for writing this text. My initial interest in BEM was sparked by its advertised modeling advantages over the domain based techniques. Ideally, the user of BEM should only have to concern himself with a boundary model. It was a mixed blessing to discover that this advantage, as of yet, had not been completely realized for all physical problems. Some domain discretization was still required in many problems, including the general Poisson equation. I was disappointed, of course, by what was available at the time, but excited at the research opportunities represented by the domain discretization problem. Chapter 5, therefore, represents an attempt at alleviating the need for explicit domain discretization in the general Poisson equation. The system used is based upon Monte Carlo quadrature and, in this author's opinion, is an excellent method for obtaining results suitable for any practical accuracy. However, there is much room left for work in the area of "domain elimination." The message reflected in Chapter 5 is that the engineer's time is too valuable to spend creating grids or generating meshes. Hopefully, this presentation will inspire more research in the area of making all boundary integral formulations strictly boundary discretizations. Chapter 6 presents some results of several example Poisson analyses based upon the considerations of Chapter 3 and Chapter 5. Finally, in Chapter 7 is presented a computer program suitable for the solution of the 2-D and axisymmetric Poisson's equation in piecewise nonhomogeneous media with linear elements. The program is constructed from the theory presented in Chapters 3 and 5, and is the one used to work the examples in Chapters 4 and 6.

Hopefully, when the entire text is read and understood, the reader will agree that BEM not only works but works well. The tone of the text necessarily take the attitude of BEM versus all other numerical methods in engineering, especially finite elements. This should not be construed as an "anti-finite element" campaign on the author's part (I routinely teach two courses in FEM), but rather a "pro-BEM" stance. BEM does have several natural tactical advantages over FEM (as the field ion example in Chapter 4 very pointedly demonstrates), and it is hoped that the "older brother," FEM, and its long time advocates, can take a few jabs without being too badly injured.

TABLE OF CONTENTS

Chapter | Page

Chapter 1

INTEGRAL EQUATIONS AND GREEN'S FUNCTIONS

1.1 Introduction

We begin by focusing our attention on the subject of integral operators. It turns out there is a close connection between differential operators (like d^2/dx^2), with which you are very familiar, and integral operators, which you may know little or nothing about. The boundary conditions needed to solve uniquely a differential equation play an important part in linking the two types of operators. The connection between them is essentially the theory of Green's functions, an area of much importance in applied mechanics, theoretical physics, and particularly in the formulation of the boundary integral equation methods. There is no way we can possibly cover the vast subject of Green's functions in this one small volume, nor is it necessary that we do so. However, it is important that the boundary element practitioner be familiar with the main ideas. Therefore, without any further discussion, we proceed to reinvent the art of solving differential equations in a different manner than that which the reader is familiar.

1.2 A New Technique for Solving Differential Equations

The most elementary differential equation would have to be

$$\frac{dy}{dx} = f(x). \tag{1.1}$$

$f(x)$ is arbitrary. The boundary condition is specified as $y(a) = y_0$. $x = a$ represents an arbitrary point where y is known a priori. We now integrate both sides of eqn. (1.1) to obtain

$$y(x) = \int f(x')dx'\Big]_{x'=x} + C. \tag{1.2}$$

Applying the boundary condition $y(a) = y_0$ we find:

$$y(a) = y_0 = \int f(x')dx'\Big]_{x'=a} + C$$

$$\Longrightarrow \quad C = y_0 - \int f(x')dx'\Big]_{x'=a}$$

$$\Longrightarrow y(x) = \int f(x')dx'\Big]_{x'=x} - \int f(x')dx'\Big]_{x=a} + y_0$$

$$y(x) = \int_a^x f(x')dx' + y_o \qquad (1.3)$$

Thus, we have achieved the feat of solving a simple differential equation by integration. However, this example is ridiculously trivial. This "new" technique implies that everytime one writes an integral, he has devised a solution to a differential equation. Let's look a bit deeper. Suppose that x lies on some closed interval [a,b]. Then we can write eqn. (1.3) as:

$$y(x) = \int_a^b f(x')\, H(x-x')dx' + y_o \qquad (1.4)$$

which $H(x-x')$ is the Heaviside step function defined as shown in Figure 1.1. Thus, eqn. (1.4) is the same as eqn. (1.3). The significance of $H(x-x')$ in the integrand is that it serves to "cut off" the integration at $x = x'$. Note that eqn. (1.4) may be written as

$$y(x) = \tilde{K}\, f(x) + y_o \qquad (1.5)$$

where $\tilde{K}$ is an <u>integral operator</u> defined by

$$\tilde{K}\, f(x) = \int_a^b H(x-x')\, f(x')dx'. \qquad (1.6)$$

The term $H(x-x')$ is called the <u>kernel</u> of the integral operator $\tilde{K}$. It should be obvious that the kernel function is dependent upon the differential operator and the boundary conditions. Herein lies one of the advantages of using integral equations as opposed to differential equations. The boundary conditions are built directly into an integral equation.

When the kernel is derived from the solution of an equation involving a differential operator, it is called the <u>Green's function</u> for the differential

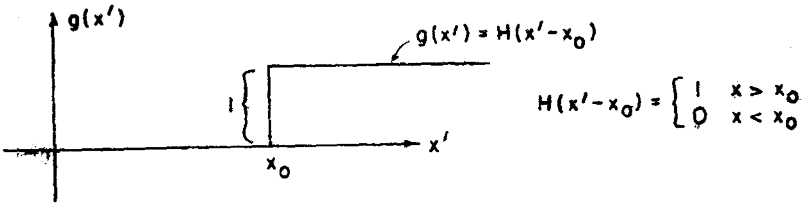

Figure 1.1 Definition of the Heaviside step function.

operator for the relevant boundary conditions. Thus, we call

$$G(x,x') = H(x-x')$$

the Green's function belonging to the differential operator d/dx for a system subject to the boundary condition $y(a) = y_o$. This particular Green's function is illustrated graphically in Figure 1.2.

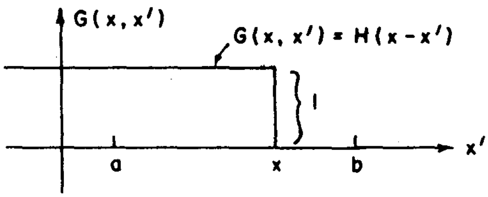

Figure 1.2 The Green's function for the operator d/dx with a prescribed left-hand boundary condition on the independent variable.

Before making any further comment, let us consider a more interesting example:

$$\frac{d^2y}{dx^2} = f(x), \tag{1.7}$$

where y(x) is subject to the boundary condition $y(x) = y_o$, and $y'(a) = \bar{y}_o$. Integrate (1.7) once to get

$$\frac{dy}{dx} = \bar{y}_o + \int_a^x f(x')dx', \tag{1.8}$$

analogous to the procedure used to get eqn. (1.3). Integrating again, and applying the boundary conditions, we get

$$y(x) = y_o + (x-a)\,\bar{y}_o + \int_a^x \int_a^{x''} f(x')dx'\,dx''. \tag{1.9}$$

[You should go through the intermediate steps.] Now, if we define x on [a,b] as before, can we find the Green's function? Yes, but with a little more trickery. Referring to Fig. 1.3, consider the differential area dx'dx" in eqn. (1.9) as a graph in the x'-x" plane. Note that eqn. (1.9) implies that we integrate over x' from a to x", and then over x" from a to x. That is, we first sweep out a horizontal strip, and then use it to sweep out the triangular area. However,

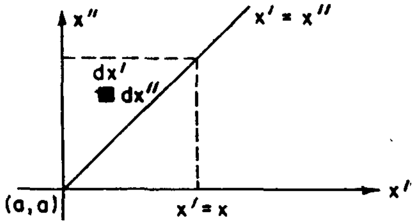

Figure 1.3 Integration in x'-x" space.

because the region is so nice, we may reverse the integration procedure and sweep out a vertical strip first, and then integrate horizontally. Switching limits this way, eqn. (1.9) becomes

$$y(x) = y_o + (x-a)\,\bar{y}_o + \int_a^x f(x')dx' \int_{x'}^x dx''$$

$$= y_o + (x-a)\,\bar{y}_o + \int_a^x (x-x')\,f(x')dx'. \quad (1.10)$$

Now, we may make our result valid over [a,b] by writing eqn. (1.10) as:

$$y(x) = y_o + (x-a)\,\bar{y}_o + \int_a^b (x-x')\,H(x-x')\,f(x')dx'. \quad (1.11)$$

Thus, we can identify the Green's function as

$$G(x,x') = (x-x')\,H(x-x') \quad (1.12)$$

for the differential operator d^2/dx^2 with the boundary conditions $y(a) = y_o$, $y'(a) = \bar{y}_o$. Thus, eqn. (11) can be written as

$$y(x) = y_o + (x-a)\,\bar{y}_o + \int_a^b G(x,x')\,f(x')dx'. \quad (1.13)$$

It is interesting to explore eqn. (1.13) a bit. Note that the homogeneous equation

$$\frac{d^2y}{dx^2} = 0$$

is covered by eqn. (1.13) if $f(x) = 0$. However, the solution given by eqn. (1.13) is trivial:

$$y(x) = y_o + (x-a)\,\bar{y}_o \quad (1.14)$$

but note the significant result that the boundary conditions for both this problem and the original nonhomogeneous equation are contained in eqn. (1.14). What we mean is that all the boundary conditions are contained in the terms without the integral! Therefore, it is almost obvious that the general solution to $d^2y/dx^2 = f(x)$ is

$$y(x) = \alpha + \beta x + \int_a^b (x-x')\, H(x-x')\, f(x')dx' \tag{1.15}$$

regardless of the boundary conditions since α and β are constants that can accommodate the boundary conditions. This is a nice feature because it allows us to determine Green's functions for different boundary conditions from a basic Green's function.

We illustrate this for equation (1.1),

$$\frac{dy}{dx} = f(x);\ y(a) = y_o \tag{1.1}$$

with an integral representation given by eqn. (1.4):

$$y(x) = \int_a^b f(x')\, H(x-x')dx' + y_o. \tag{1.4}$$

The general solution of $dy/dx = f(x)$ on the interval $[a,b]$ is:

$$y(x) = \int_a^b f(x')\, H(x-x')dx' + A. \tag{1.16}$$

We now examine the effect of changing the boundary condition to $y(b) = y_1$ instead of $y(a) = y_o$. Eqn. (1.16) gives

$$y(b) = y_1 = \int_a^b f(x')\, H(b-x')dx' + A$$

$$A = y_1 - \int_a^b f(x')\, H(b-x')dx' \tag{1.17}$$

We note that x' is always less than b, and thus eqn. (1.17) reduces to:

$$A = y_1 - \int_a^b f(x')dx'. \tag{1.18}$$

Therefore:

$$y(x) = y_1 + \int_a^b f(x')\, H(x-x')dx' - \int_a^b f(x')dx'$$

$$= y_1 + \int_a^b f(x')\, [H(x-x') - 1]dx'.$$

The Green's function is $G(x,x') = H(x-x') - 1$, or

$$G(x,x') = \begin{cases} -1 \text{ for } x < x' \\ 0 \text{ for } x > x'. \end{cases}$$

Note the way that the form of the boundary conditions dictates the form of the Green's function. Also notice that this Green's function is symmetric. That is, $G(x,x') = G(x',x)$, as is illustrated in Figure 1.4. This feature is frequently found in Green's functions that apply to physical problems, and is a subject that we will discuss later.

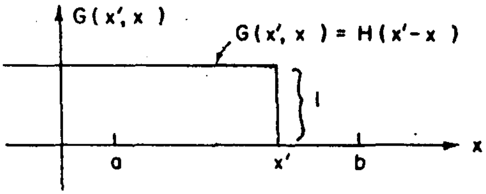

Figure 1.4 $G(x',x)$ as opposed to $G(x,x')$ shown in Fig. 1.2. Note that the functions are identical.

Homework Problem #1.1:

Use equation (1.15) to derive the Green's function corresponding to $d^2y/dx^2 = f(x)$ for $a < x < b$ with boundary conditions $y(a) = y_o$ and $y(b) = y_1$.

At this point you may have severe questions concerning validity of the Green's function technique of solving the differential equations of engineering. At the risk of being repetitious, let us reiterate that once the Green's function for the particular boundary conditions is determined, the problem is effectively solved. In fact, an applied mathematician would probably take an expression like eqn. (1.11) as final. An engineer, on the other hand, would prefer to have an explicit solution. This may be done analytically if $f(x)$ is conveniently simple. If the integral cannot be performed exactly, most likely it can be done numerically using standard numerical integration techniques. The numerical alternative allows the possibility of $f(x)$ being defined by discrete tabular data instead of an explicit function. Such is frequently the case in experimental work. Before