BRIAN D. O. ANDERSON
JOHN B. MOORE

Linear Optimal Control

BRIAN D. O. ANDERSON
Professor of Electrical Engineering
University of Newcastle
New South Wales, Australia

JOHN B. MOORE
Associate Professor
Department of Electrical Engineering
University of Newcastle

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PREFACE

Despite the development of a now vast body of knowledge known as modern control theory, and despite some spectacular applications of this theory to practical situations, it is quite clear that much of the theory has yet to find application, and many practical control problems have yet to find a theory which will successfully deal with them. No book of course can remedy the situation at this time. But the aim of this book is to construct one of many bridges that are still required for the student and practicing control engineer between the familiar classical control results and those of modern control theory. It attempts to do so by consistently adopting the viewpoints that

1. many modern control results have interpretation in terms of classical control results;
2. many modern control results do have practical engineering significance, as distinct from applied mathematical significance.

As a consequence, linear systems are very heavily emphasized, and, indeed, the discussion of nonlinear systems is essentially restricted to two classes: systems which should be linear, but unhappily are not; and systems which are linear but for the intentional use of relays. Also as a consequence of this approach, discussion of some results deemed fundamental in the general theory of optimal control has been kept to the barest minimum, thereby allowing emphasis on those particular optimal control results having application to linear systems. It may therefore seem strange to present a book on optimal control which does not discuss the Pontryagin Maximum Principle, but it is nonetheless consistent with the general aims of the book.

Although the selection of the material for the book has not been governed by the idea of locating the optimal control theory of linear systems within the broader framework of optimal control theory per se, it has been governed by the aim of presenting results of linear optimal control theory interesting from an engineering point of view, consistent with the ability of students to follow the material. This has not meant restricting the choice of material presented to that covered in other books; indeed a good many of the ideas discussed have appeared only in technical papers.

For the most part, continuous time systems are treated, and a good deal more of the discussion is on time-invariant than is on time-varying systems. Infinite-time optimization problems for time-varying systems involve concepts such as uniform complete controllability, which the authors consider to be in the nature of advanced rather than core material, and accordingly discussion of such material is kept to a minimum. For completeness, some mention is also made of discrete-time systems, but it seemed to us that any extended discussion of discrete-time systems would involve undue repetition.

The text is aimed at the first or later year graduate student. The background assumed of any reader is, first, an elementary control course, covering such notions as transfer functions, Nyquist plots, root locus, etc., second, an elementary introduction to the state-space description of linear systems and the dual notions of complete controllability and complete observability, and third, an elementary introduction to linear algebra. However, exposure to a prior or concurrent course in optimal control is not assumed. For students who have had a prior course, or are taking concurrently a course in the general theory of optimal control or a specific aspect of the discipline such as time-optimal systems, a course based on this book will still provide in-depth knowledge of an important area of optimal control.

Besides an introductory chapter and a final chapter on computational aspects of optimal linear system design, the book contains three major parts. The first of these outlines the basic theory of the linear regulator, for time-invariant and time-varying systems, emphasizing the former. The actual derivation of the optimal control law is via the Hamilton-Jacobi equation which is introduced using the Principle of Optimality. The infinite-time problem is considered, with the introduction of exponential weighting in the performance index used for time-invariant design as a novel feature. The second major part of the book outlines the engineering properties of the regulator, and attempts to give the reader a feeling for the use of the optimal linear regulator theory as a design tool. Degree of stability, phase and gain margin, tolerance of time delay, effect of nonlinearities, introduction of relays, design to achieve prescribed closed-loop poles, various sensitivity problems, state estimation and design of practical controllers are all considered. The third major part of the book discusses extensions to the servomechanism problem, to the situation where the derivative of the control may be limited (leading to dynamic feedback controllers) or the control itself may be limited in amplitude (leading to feedback controllers containing relays), and to recent results on output, as distinct from state feedback. Material on discrete time systems and additional material on time-varying continuous systems is also presented. The final part of the book, consisting of one chapter only, discusses approaches to the solution of Riccati equations, including approximate solution procedures based on singular perturbation theory. Appendices summarizing matrix theory and linear system theory results relevant to the material of the book are also included.

Readers who have been introduced to the regulator problem elsewhere may find section 3 of chapter 3 a convenient starting point, unless review of the earlier material is required.

We would like to emphasize that the manuscript was compiled as a truly joint effort; it would be difficult to distinguish completely who wrote what section and whose ideas were involved at each point in the development of the material. Both of us were surprised at the fact that working together we could achieve far more than either of us working independently, and we are thankful for the personal enrichment to our lives from the experience of working together.

In listing acknowledgments, our former teachers Robert Newcomb and Dragoslav Ŝiljak come immediately to mind as do our graduate students Peter Moylan and Konrad Hitz. In knowing these people each of us has learned that a teacher-student relationship can be infinitely more worthwhile than the usual connotation of the words implies. We appreciate the direct help given by our students as we also appreciate the work done by Brian Thomas in drafting the various diagrams, and Sue Dorahy, Pam O'Sullivan and Lorraine Youngberry for typing the manuscript. We are happy to acknowledge the financial support of our research by the Australian Research Grants Committee. We mention our families—Dianne and Elizabeth Anderson, and Jan and Kevin Moore because they are a part of us. We mention all these various names along with our own in recognition of the world of people out of whom the world of ideas is born and for whom it exists.

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JOHN B. MOORE

CONTENTS

PART I

INTRODUCTION

CHAPTER 1

INTRODUCTION

1.1 LINEAR OPTIMAL CONTROL

The methods and techniques of what is now known as "classical control" will be familiar to most readers. In the main, the systems or plants that can be considered by using classical control ideas are linear, time invariant, and have a single input and a single output. The primary aim of the designer using classical control design methods is to stabilize a plant, whereas secondary aims may involve obtaining a certain transient response, bandwidth, steady state error, and so on. The designer's methods are a combination of analytical ones (e.g., Laplace transform, Routh test), graphical ones (e.g., Nyquist plots, Nichols charts), and a good deal of empirically based knowledge (e.g., a certain class of compensator works satisfactorily for a certain class of plant). For high-order systems, multiple-input systems, or systems that do not possess the properties usually assumed in the classical control approach, the designer's ingenuity is generally the limiting factor in achieving a satisfactory design.

Two of the main aims of modern, as opposed to classical, control are to deempiricize control system design and to present solutions to a much wider class of control problems than classical control can tackle. One of the major ways modern control sets out to achieve these aims is by providing an array of analytical design procedures that lessen the load of the design task on the designer's ingenuity and locate more of the load upon his mathematical ability and on the computational machines used in actually carrying out the design.

Optimal control is one particular branch of modern control that sets out to provide analytical designs of a specially appealing type. The system which is the end result of an optimal design is not supposed merely to be stable, have a certain bandwidth, or satisfy any one of the desirable constraints associated with classical control, but it is supposed to be the *best* possible system of a particular type—hence, the word optimal. If it is both optimal and possesses a number of the properties that classical control suggests are desirable, so much the better.

Linear optimal control is a special sort of optimal control. The plant that is controlled is assumed linear, and the controller, the device which generates the optimal control, is constrained to be linear. That is, its output, the optimal control, is supposed to depend linearly on its input, which will consist of quantities derived from measurements on the plant. Of course, one may well ask: Why linear optimal control, as opposed simply to optimal control? A number of justifications may be advanced—for example, many engineering plants are linear prior to addition of a controller to them; a linear controller is simple to implement physically, and will frequently suffice.

Other advantages of optimal control, when it is specifically linear, follow.

1. Many optimal control problems do not have computable solutions, or they have solutions that may only be obtained with a great deal of computing effort. By contrast, nearly all linear optimal control problems have readily computable solutions.
2. Linear optimal control results may be applied to nonlinear systems operating on a small signal basis. More precisely, suppose an optimal control has been developed for some nonlinear system with the assumption that this system will start in a certain initial state. Suppose, however, that the system starts in a slightly different initial state, for which there exists some other optimal control. Then a first approximation to the difference between the two optimal controls may normally be derived, if desired, by solving a linear optimal control problem (with all its attendant computational advantages). This holds independently of the criterion for optimality for the nonlinear system. (Since this topic will not be discussed anywhere in this book, we list the two references [1] and [2] that outline this important result.†)
3. The computational procedures required for linear optimal design may often be carried over to nonlinear optimal problems. For example,

†References are located at the end of each chapter.

the nonlinear optimal design procedures based on the theory of the second variation [1–3] and quasilinearization [3, 4] consist of computational algorithms replacing the nonlinear problem by a sequence of linear problems.

4. Linear optimal control designs turn out to possess a number of properties, other than simply optimality, which classical control suggests are attractive. Examples of such properties are good gain margin and phase margin, and good tolerance of nonlinearities. This latter property suggests that controller design for nonlinear systems may sometimes be achieved by designing with the assumption that the system is linear (even though this may not be a good approximation), and by relying on the fact that an optimally designed linear system can tolerate nonlinearities—actually quite large ones—without impairment of all its desirable properties. Hence, linear optimal design methods are in some ways applicable to nonlinear systems.
5. Linear optimal control provides a framework for the unified study of the control problems studied via classical methods. At the same time, it vastly extends the class of systems for which control designs may be achieved.

1.2 ABOUT THIS BOOK IN PARTICULAR

This is not a book on optimal control, but a book on linear optimal control. Accordingly, it reflects very little of the techniques or results of general optimal control. Rather, we study a basic problem of linear optimal control, the "regulator problem," and attempt to relate mathematically all other problems discussed to this one problem. If the reader masters the mathematics of the regulator problem, he should find most of the remainder of the mathematics relatively easy going. (Those familiar with the standard regulator and its derivation may bypass Chapter 2, Sec. 2.1 through Chapter 3, Sec. 3.3. Those who wish to avoid the mathematics leading to regulator results in a first reading may bypass Chapter 2, Sec. 2.2 through Chapter 3, Sec. 3.3.)

The fact that we attempt to set up mathematical relations between the regulator problem and the other problems considered does not mean that we seek, or should seek, physical or engineering relations between the regulator problem and other problems. Indeed, these will not be there, and even the initial mathematical statements of some problems will often not suggest their association with the regulator problem.

We aim to analyze the engineering properties of the solution to the problems presented. We thus note the various connections to classical control results and ideas, which, in view of their empirical origins, are often best for assessing a practical design, as distinct from arriving at this design.

1.3 PART AND CHAPTER OUTLINE

In this section, we briefly discuss the breakdown of the book into parts and chapters. There are five parts, listed below with brief comments.

Part I—Introduction. This part is simply the introductory first chapter.

Part II—Basic theory of the optimal regulator. These chapters serve to introduce the linear regulator problem and to set up the basic mathematical results associated with it. Chapter 2 sets up the problem, by translating into mathematical terms the physical requirements on a regulator. It introduces the Hamilton–Jacobi equation as a device for solving optimal control problems, and then uses this equation to obtain a solution for problems where performance over a finite (as opposed to infinite) time interval is of interest. The infinite-time interval problem is considered in Chapter 3, which includes stability properties of the optimal regulators. Chapter 4 shows how to achieve a regulator design with a prescribed degree of stability.

Part III—Properties and application of the optimal regulator. The aim of this part is twofold. First, it derives and discusses a number of engineering properties of the linear optimal regulator, and, second, it discusses the engineering implementation of the regulator. The main purpose of Chapter 5 is to derive some basic frequency domain formulas and to use these to deduce from Nyquist plots properties of optimal systems involving gain margin, etc. In this chapter, the problem is also considered of designing optimal systems with prescribed closed-loop poles. In Chapter 6, an examination is made of the effect of introducing nonlinearities, including relays, into optimal systems. The main point examined is the effect of the nonlinearities on the system stability. Chapter 7 is mainly concerned with the effect of plant parameter variations in optimal systems, and studies the effect using modern control ideas as well as the classical notion of the return difference. There is also further discussion in Chapter 7 of the design of optimal systems with prescribed closed-loop poles. Chapter 8 is devoted to the problem of state estimation; implementation of optimal control laws generally requires the feeding back of some function of the plant state vector, which may need to be estimated from the plant input and output if it is not directly measurable. The discussions of Chapter 8 include estimators that operate optimally in the presence of noise, the design of such estimators being achieved via solution of an optimal regulator problem. The purpose of Chapter 9 is to tie the estimation procedures of Chapter 8 with the optimal control results of earlier chapters so as to achieve controllers of some engineering utility. Attention is paid to simplification of the structure of these controllers.

Part IV—Extensions to more complex problems. In this part, the aim is to use the regulator results to solve a number of other linear optimal control problems of engineering interest. Chapter 10 considers problems resulting in controllers using proportional-plus-integral state feedback. Chapter 11 considers various versions of the classical servomechanism problem. Chapter 12 considers problems when there is an upper bound on the magnitude of the control; the controllers here become dual mode, with one mode—the linear one—computable using the regulator theory. Next, Chapter 13 considers problems where only the plant output is available for use in a nondynamic controller, as well as other optimal problems that include controller constraints. Such problems are often referred to as suboptimal problems. Chapter 14 contains a very brief discussion of discrete time systems, and continuous time-varying systems on an infinite-time interval.

Part V—Computational aspects. This part—Chapter 15—discusses some of the computational difficulties involved in carrying out an optimal control design. Various techniques are given for finding transient and steady state solutions to an equation, the matrix Riccati equation, occurring constantly in linear design. Approximate solutions are discussed, as well as a description of situations in which these approximate solutions are applicable.

Appendices. Results in matrix theory and linear system theory relevant to the material in the book are summarized in the appendices.

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