Lecture Notes in Control and Information Sciences

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Two-Dimensional Linear Systems

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Preface

A growing interest has been developed over the past few years in problems involving signals and systems that depend on more than one variable. These multidimensional signals and systems have been studied in relation to several modern engineering fields such as multidimensional digital filtering, multivariable network realizability, multidimensional system synthesis digital picture processing, seismic data processing, X-ray image enhancement, the enhancement and analysis of aerial photographs for detection of forest fires or crop damage, the analysis of satellite weather photos, image deblurring, etc. Most of the major results concerning the multidimensional signals and systems are developed for two-dimensional /2-D/ cases.

These results may be grouped as follows.

1. 2-D systems and filters. The 2-D linear shift invariant systems are described by a convolution of the input and the unit impulse response. Some of the problems already investigated refer to the questions of recursibility, stability and limit cycles.
2. 2-D state-space models. Based on the state-space description several properties of 2-D systems such as controllability, observability, canonical forms, minimality, etc. have been investigated.
3. 2-D image processing, random fields and space-time processing. These problems have drawn considerable attention and have shown great potential for practical applications such as X-ray image enhancement, image deblurring, weather prediction, seismic data analysis, radar and sonar array processing, etc.
4. 2-D feedback design techniques. These problems refer to the general area of developing feedback design techniques so that the closed-loop system has pre-assigned desirable characteristics. The eigenvalue assignment exact model-matching, transfer function factorization, minimum energy control, observers have been considered

in many papers.

The main objective of this monograph is to present recent developments in 2-D linear system theory.

The monograph is organised as follows.

Chapter 1 presents Roesser's model, Attasi's model and two Fornasini-Marchesini's models. The transition matrices for the models are defined and the general response formulas are given.

The transfer function matrix, the realization problem and the separability of transfer function matrices are considered in Chapter 2.

Different notions of the controllabity, observability and reachability are described in Chapter 3. The minimum energy control of 2-D systems is also considered.

Chapter 4 gives definitions and stability tests for 2-D systems described by the transfer function matrices and the state equations.

The stabilization problems are also considered. Some new methods concerning eigenvalue assignment for 2-D and 3-D linear systems are given in Chapter 5. The asymptotic and deadbeat observers, the exact model matching and the decoupling are considered in Chapter 6.

Finally, Chapter 7 presents some new results concerning deadbeat control and deadbeat servo problems.

An Appendix of basic definitions, theorems and computational algorithms has been included for the sake of greater comprehensiveness.

The monograph is addressed to graduate students specializing in control, scientists and engineers engaged in control system research and development and mathematicians interested in control problems.

I wish to thank dr B.Eichsteadt and dr M.Kocięcki for their valuable remarks, suggestions and comments.

T.Kaczorek

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1 STATE-SPACE MODELS AND RESPONSE FORMULAE

1.1 STATE-SPACE MODELS OF TWO-DIMENSIONAL LINEAR SYSTEMS

Roesser's model.

Roesser's model /RM/ is defined by the equations [13]

$$\begin{bmatrix} x^h/i+1,j/ \\ x^v/i,j+1/ \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h/i,j/ \\ x^v/i,j/ \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u/i,j/ \qquad /1.1/$$

$$y/i,j/ = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x^h/i,j/ \\ x^v/i,j/ \end{bmatrix} + D\, u/i,j/ \qquad (i,j \geq 0) \qquad /1.2/$$

where i is an integer-valued vertical coordinate,
j is an integer-valued horizontal coordinate,
$x^h/i,j/ \in R^{n_1}$ is the horizontal state vector,
$x^v/i,j/ \in R^{n_2}$ is the vertical state vector,
$u/i,j/ \in R^m$ is the input vector,
$y/i,j/ \in R^l$ is the output vector,
$A_{11}, A_{12}, A_{21}, A_{22}, B_1, B_2, C_1, C_2, D$ are real matrices of appropriate dimensions.

Boundary conditions for /1.1/ are given by

$$x^h/0,j/, \quad x^v/i,0/ \quad \text{for } i,j = 0,1,2,\ldots \qquad /1.1a/$$

Introducing the matrices and vectors

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix},$$

$$x' = \begin{bmatrix} x^h/i+1,j/ \\ x^v/i,j+1/ \end{bmatrix}, \quad x = \begin{bmatrix} x^h/i,j/ \\ x^v/i,j/ \end{bmatrix}, \quad u = u/i,j/, \quad y = y/i,j/$$

we can rewrite /1.1/ and /1.2/ in the form

$$x' = Ax + Bu \qquad /1.1'/$$

$$y = Cx + Du \qquad /1.2'/$$

Example 1.1

Consider the equation [12]

$$\frac{\partial T/x,t/}{\partial x} = -\frac{\partial T/x,t/}{\partial t} - T/x,t/ + U/t/ \qquad /1.3/$$

with initial and boundary conditions

$$T/x,0/ = f_1/x/, \quad T/0,t/ = f_2/t/ \qquad /1.3a/$$

where $T/x,t/$ is an unknown function (usually the temperature) at x(space) $\in [0, x_f]$ and t(time) $\in [0, \infty]$, $U/t/$ is a given force function and $f_1/x/$, $f_2/t/$ are given functions.

The equation /1.3/ describes some thermal processes, for example in chemical reactors, heat exchangers and pipe furnaces /Fig. 1.1/.

Taking

$$T/i,j/ = T/i\Delta x, j\Delta t/, \quad U/j/ = U/j\Delta t/,$$

$$\frac{\partial T/x,t/}{\partial t} \cong \frac{T/i,j+1/ - T/i,j/}{\Delta t} \qquad \frac{\partial T/x,t/}{\partial x} \cong \frac{T/i,j/ - T/i-1,j/}{\Delta x}$$

we can write /1.3/ in the form

$$T/i,j+1/ = a_1 T/i,j/ + a_2 T/i-1,j/ + bU/j/ \qquad /1.4/$$

where

$$a_1 = 1 - \frac{\Delta t}{\Delta x} - \Delta t, \quad a_2 = \frac{\Delta t}{\Delta x}, \quad b = \Delta t.$$

If we define

$$x^h/i,j/ = T/i-1,j/ \quad \text{and} \quad x^v/i,j/ = T/i,j/$$

then from /1.4/ we obtain the Roesser's model

$$\begin{bmatrix} x^h/i+1,j/ \\ x^v/i,j+1/ \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix} \begin{bmatrix} x^h/i,j/ \\ x^v/i,j/ \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} U/j/ \qquad /1.5/$$

Example 1.2

Consider the equations

$$\frac{\partial u/x,t/}{\partial x} = L\frac{\partial i/x,t/}{\partial t}, \quad \frac{\partial i/x,t/}{\partial x} = C\frac{\partial u/x,t/}{\partial t} \qquad /1.6/$$

which describe voltage $u/x,t/$ and current $i/x,t/$ at x(space) $\in [0,1]$ and t(time) $\in [0,\infty]$ in a long transmission line (Fig. 1.2).

The initial and boundary conditions are given by

$$u/x,0/ = U/x/, \quad i/x,0/ = I/x/$$
$$u/0,t/ = U_1/t/, \quad u/1,t/ = U_2/t/ \qquad /1.6a/$$

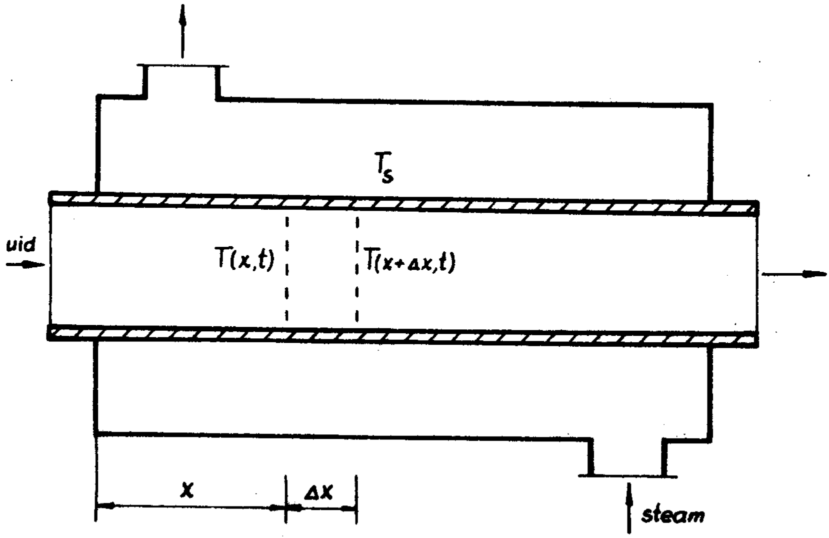

Fig. 1.1 Heat exchanger.

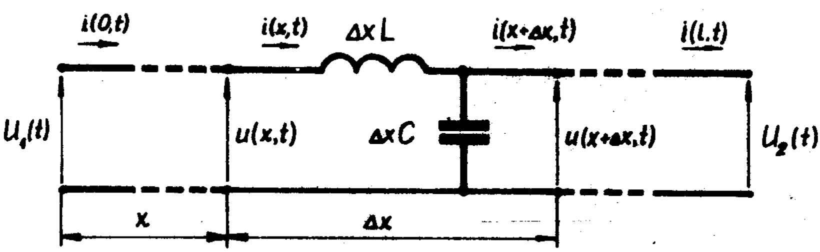

Fig. 1.2 Transmission line.

The equations /1.6/ can be rewritten in the form

$$\frac{\partial}{\partial t}\begin{bmatrix} u/x,t/ \\ i/x,t/ \end{bmatrix} = A \frac{\partial}{\partial x}\begin{bmatrix} u/x,t/ \\ i/x,t/ \end{bmatrix} \qquad /1.7/$$

where

$$A = \begin{bmatrix} 0 & \frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \qquad /1.8/$$

Let us define

$$\begin{bmatrix} u/x,t/ \\ i/x,t/ \end{bmatrix} = T\begin{bmatrix} \bar{u}/x,t/ \\ \bar{i}/x,t/ \end{bmatrix} \qquad /1.9/$$

where

$$T = \begin{bmatrix} 1 & \frac{L}{\sqrt{LC}} \\ \frac{C}{\sqrt{LC}} & -1 \end{bmatrix}$$

is the matrix whose columns are the eigenvectors of /1.8/.
It is easy to check that

$$\frac{\partial}{\partial t}\begin{bmatrix} \bar{u}/x,t/ \\ \bar{i}/x,t/ \end{bmatrix} = \bar{A} \frac{\partial}{\partial x}\begin{bmatrix} \bar{u}/x,t/ \\ \bar{i}/x,t/ \end{bmatrix} \qquad /1.10/$$

where

$$\bar{A} = T^{-1}AT = \begin{bmatrix} \frac{1}{\sqrt{LC}} & 0 \\ 0 & -\frac{1}{\sqrt{LC}} \end{bmatrix}$$

To find the Roesser's model for /1.10/ we can apply the procedure used for /1.3/.

Example 1.3

Consider the Darboux equation [12]

$$\frac{\partial^2 s/x,t/}{\partial x\, \partial t} = a_1 \frac{\partial s/x,t/}{\partial t} + a_2 \frac{\partial s/x,t/}{\partial x} + a_o\, s/x,t/ + b\, f/x,t/ \qquad /1.11/$$

with the initial and boundary conditions

$$s/x,0/ = S_1/x/, \quad s/0,t/ = S_2/t/ \qquad /1.11a/$$

where s/x,t/ is an unknown function at x(space) $\in [0,x_f]$ and t(time) $\in$ $\in [0,\infty]$, a_o, a_1, a_2 and b are real coefficients, f/x,t/ is a given input function and $S_1/x/$, $S_2/t/$ are given functions.

The equation /1.11/ describes some linear processes of gas absorption, water stream heating and air drying.

Let us define

$$r/x,t/ = \frac{\partial s/x,t/}{\partial t} - a_2\, s/x,t/ \qquad /1.12/$$

Using /1.12/ we can transform /1.11/ into an equivalent system of first order differential equations of the form

$$\begin{bmatrix} \frac{\partial r/x,t/}{\partial x} \\ \frac{\partial s/x,t/}{\partial t} \end{bmatrix} = \begin{bmatrix} a_1 & a_1 a_2 + a_0 \\ 1 & a_2 \end{bmatrix} \begin{bmatrix} r/x,t/ \\ s/x,t/ \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} f/x,t/ \qquad /1.13/$$

From /1.12/ and /1.11a/ we have

$$r/0,t/ = \frac{\partial s/x,t/}{\partial t}\Big|_{x=0} - a_2\, s/0,t/ = \frac{d\, S_2/t/}{dt} - a_2\, S_2/t/ =$$

$$= R/t/ \qquad /1.14/$$

Taking

$$r/i,j/ = r/i\Delta x, j\Delta t/$$

$$\frac{\partial r/x,t/}{\partial x} \simeq \frac{r/i+1,j/ - r/i,j/}{\Delta x}, \quad \frac{\partial s/x,t/}{\partial t} \simeq \frac{s/i,j+1/ - s/i,j/}{\Delta t}$$

we obtain from /1.13/ the following Roesser's model

$$\begin{bmatrix} r/i+1,j/ \\ s/i,j+1/ \end{bmatrix} = \begin{bmatrix} 1+a_1\Delta x & (a_1a_2+a_0)\Delta x \\ \Delta t & 1 + a_2\Delta t \end{bmatrix} \begin{bmatrix} r/i,j/ \\ s/i,j/ \end{bmatrix} + \begin{bmatrix} b\,\Delta x \\ 0 \end{bmatrix} f/i,j/ \qquad /1.15/$$

with boundary conditions

$$r/0,j/ = R/j\,\Delta t/ \qquad /1.15a/$$

$$s/i,0/ = S_1/i\,\Delta x/$$

Attasi's model.

Attasi's model /AM/ is defined by the equations [1, 2]

$$\bar{x}/i+1,j+1/ = \bar{A}_1\bar{x}/i+1,j/ + \bar{A}_2\bar{x}/i,j+1/ - \bar{A}_1\bar{A}_2\bar{x}/i,j/ + \bar{B}u/i,j/ \quad /1.16/$$

$$y/i,j/ = \bar{C}\bar{x}/i,j/ \qquad \text{with } \bar{A}_1\bar{A}_2 = \bar{A}_2\bar{A}_1, \ (i,j \geq 0) \quad /1.17/$$

where i, j are integer-valued vertical and horizontal coordinates, respectively,
$\bar{x}/i,j/ \in R^n$ is the local state vector at /i,j/,
$u/i,j/ \in R^m$ is the input vector,
$y/i,j/ \in R^l$ is the output vector,
$\bar{A}_1$, $\bar{A}_2$, $\bar{B}$, $\bar{C}$ are real matrices of appropriate dimensions.

Boundary conditions for /1.16/ are given by

$$\bar{x}/i,0/, \ \bar{x}/0,j/ \ \text{for } i,j = 0,1,2,\ldots \quad /1.17a/$$

Fornasini - Marchesini's models.

The first Fornasini - Marchesini's model /F-MM I/ is defined by the equations [4]

$$\hat{x}/i+1,j+1/ = \hat{A}_0\hat{x}/i,j/ + \hat{A}_1\hat{x}/i+1,j/ + \hat{A}_2\hat{x}/i,j+1/ + \hat{B}u/i,j/ \quad /1.18/$$

$$y/i,j/ = \hat{C}\hat{x}/i,j/ \qquad (i,j \geq 0) \quad /1.19/$$

where i, j are integer-valued vertical and horizontal coordinates, respectively,
$\hat{x}/i,j/ \in R^n$ is the local state vector at /i,j/,
$u/i,j/ \in R^m$ is the input vector,
$y/i,j/ \in R^l$ is the output vector,
$\hat{A}_0$, $\hat{A}_1$, $\hat{A}_2$, $\hat{B}$, $\hat{C}$ are real matrices of appropriate dimensions.

Boundary conditions for /1.18/ are given by

$$\hat{x}/i,0/, \ \hat{x}/0,j/ \ \text{for } i,j = 0,1,2,\ldots \quad /1.18a/$$

The second Fornasini - Marchesini's model /F - MM II / is defined by the equations [3]

$$x/i+1,j+1/ = A_1x/i,j+1/ + A_2x/i+1,j/ + B_{01}u/i+1,j/ + B_{10}u/i,j+1/ \quad /1.20/$$

$$y/i,j/ = Cx/i,j/ + Du/i,j/ \qquad (i,j \geq 0) \qquad /1.21/$$

where i, j are integer-valued vertical and horizontal coordinates, respectively,

$x/i,j/ \in R^n$ is the local state vector at /i,j/,

$u/i,j/ \in R^m$ is the input vector,

$y/i,j/ \in R^l$ is the output vector,

A_1, A_2, B_{10}, B_{01}, C, D are real matrices of appropriate dimensions.

Boundary conditions for /1.20/ are given by

$$x/i,0/, \quad x/0,j/ \quad \text{for} \quad i,j = 1,2,\ldots \qquad /1.20a/$$

1.2 RELATIONS BETWEEN THE MODELS

From comparison of /1.16/ and /1.18/ it follows that AM is a special case of F-MM I for $\hat{A}_0 = -\bar{A}_1\bar{A}_2 = -\bar{A}_2\bar{A}_1$.

Let us define

$$x^h/i,j/ = \hat{x}/i,j+1/ - \hat{A}_1\hat{x}/i,j/ \quad \text{and} \quad x^v/i,j/ = \hat{x}/i,j/ .$$

Taking into account /1.18/ we can write

$$x^h/i+1,j/ = \hat{A}_0x^v/i,j/ + \hat{A}_2\left[x^h/i,j/ + \hat{A}_1x^v/i,j/\right] + \hat{B}u/i,j/ =$$

$$= \hat{A}_2x^h/i,j/ + \left[\hat{A}_0+\hat{A}_2\hat{A}_1\right]x^v/i,j/ + \hat{B}u/i,j/$$

and

$$x^v/i,j+1/ = x^h/i,j/ + \hat{A}_1x^v/i,j/.$$

Hence

$$\begin{bmatrix} x^h/i+1,j/ \\ x^v/i,j+1/ \end{bmatrix} = \begin{bmatrix} \hat{A}_2 & \hat{A}_0+\hat{A}_2\hat{A}_1 \\ I_n & \hat{A}_1 \end{bmatrix}\begin{bmatrix} x^h/i,j/ \\ x^v/i,j/ \end{bmatrix} + \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix}u/i,j/$$

and

$$y/i,j/ = \begin{bmatrix} 0 & \hat{C} \end{bmatrix}\begin{bmatrix} x^h/i,j/ \\ x^v/i,j/ \end{bmatrix}.$$

Thus, F-MM I can be recasted in RM with

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} \hat{A}_2 & \hat{A}_0+\hat{A}_2\hat{A}_1 \\ I_n & \hat{A}_1 \end{bmatrix}, \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} C_1 & C_2 \end{bmatrix} = \begin{bmatrix} 0 & \hat{C} \end{bmatrix}, \quad D = 0 \qquad /1.22/$$

It is easy to show that if $A_{21} = I_n$, $B_2 = 0$ and $C_1 = 0$ then RM can also be recasted in F-MM I with

$$\hat{A}_0 = A_{12} - A_{11}A_{22}, \quad \hat{A}_1 = A_{22}, \quad \hat{A}_2 = A_{11}, \quad \hat{B} = B_1, \quad \hat{C} = C_2 .$$

In particular case AM can be recasted in RM with

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} \bar{A}_2 & 0 \\ I_n & \bar{A}_1 \end{bmatrix}, \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} \bar{B} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} C_1 & C_2 \end{bmatrix} = \begin{bmatrix} 0 & \bar{C} \end{bmatrix}, \quad D = 0$$

and if $A_{12} = 0$, $A_{21} = I_n$, $A_{11}A_{22} = A_{22}A_{11}$, $B_2 = 0$ and $C_1 = 0$ then RM can also be recasted in AM with

$$\bar{A}_1 = A_{22}, \quad \bar{A}_2 = A_{11}, \quad \bar{B} = B_1, \quad \bar{C} = C_2 .$$

It will be shown that RM is a particular case of F-MM II .
Defining

$$x/i,j/ = \begin{bmatrix} x^h/i,j/ \\ x^v/i,j/ \end{bmatrix}$$

we can write /1.1/ and /1.2/ in the form

$$x/i+1,j+1/ = \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} x/i+1,j/ + \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} x/i,j+1/ + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u/i+1,j/ + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u/i,j+1/$$

and

$$y/i,j/ = \begin{bmatrix} C_1 & C_2 \end{bmatrix} x/i,j/ + Du/i,j/.$$

Thus, RM is a particular case of F-MM II with

$$A_2 = \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad A_1 = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}, \quad B_{01} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad B_{10} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \qquad /1.23/$$

Note that /1.18/ and /1.19/ can be written in the form

$$\begin{bmatrix} \hat{x}/i+1,j+1/ \\ \hat{x}/i+1,j/ \\ u/i+1,j/ \end{bmatrix} = \begin{bmatrix} \hat{A}_1 & 0 & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}/i+1,j/ \\ \hat{x}/i+1,j-1/ \\ u/i+1,j-1/ \end{bmatrix} + \begin{bmatrix} \hat{A}_2 & \hat{A}_0 & \hat{B} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}/i,j+1/ \\ \hat{x}/i,j/ \\ u/i,j/ \end{bmatrix} +$$

$$+ \begin{bmatrix} 0 \\ 0 \\ I_m \end{bmatrix} u/i+1,j/ + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} u/i,j+1/$$

$$y/i,j/ = \begin{bmatrix} \hat{C} & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}/i,j/ \\ \hat{x}/i,j-1/ \\ u/i,j-1/ \end{bmatrix}.$$

Assuming the vector

$$x/i,j/ = \begin{bmatrix} \hat{x}/i,j/ \\ \hat{x}/i,j-1/ \\ u/i,j-1/ \end{bmatrix}$$

as local state vector of F-MM II it is easy to see that /1.18/ and /1.19/ can be rewritten in the form /1.20/ and /1.21/ with

$$A_2 = \begin{bmatrix} \hat{A}_1 & 0 & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} \hat{A}_2 & \hat{A}_0 & \hat{B} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_{01} = \begin{bmatrix} 0 \\ 0 \\ I_m \end{bmatrix}, \quad B_{10} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} \hat{C} & 0 & 0 \end{bmatrix}, \quad D = 0.$$

Therefore F-MM I can be embedded in F-MM II .

<u>1.3 TRANSITION MATRIX AND GENERAL RESPONSE FORMULA FOR ROESSER´S MODEL</u>

The following partial ordering is used for integer pairs

$/h,k/ \leqslant /i,j/$ if and only if $h \leqslant i$ and $k \leqslant j$;

$/h,k/ = /i,j/$ if and only if $h = i$ and $k = j$;

$/h,k/ < /i,j/$ if and only if $/h,k/ \leqslant /i,j/$ and $/h,k/ \neq /i,j/$.

The transition /state - transition/ matrix $A^{i,j}$ for

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{ii} \in R^{n_i \times n_i} \quad /i = 1,2/ \qquad /1.24/$$

is defined as follows [13]:

1° $A^{0,0} = I$ /the identity matrix/

$$2° \quad A^{1,0} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}, \quad A^{0,1} = \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} \qquad /1.25/$$

3° $A^{i,j} = A^{1,0}A^{i-1,j} + A^{0,1}A^{i,j-1}$ for $/i,j/ > /0,0/$

4° $A^{i,j} = 0$ /the zero matrix/ for $i < 0$ or $j < 0$

From /1.25/ for $j = 0$ we have

$$A^{i,0} = A^{1,0}A^{i-1,0} + A^{0,1}A^{i,-1} = A^{1,0}A^{i-1,0} \qquad /1.26/$$

because $A^{i,-1} = 0$. From /1.26/ it follows that

$$A^{i,0} = \left(A^{1,0}\right)^i \quad \text{for} \quad i = 1,2,\ldots \qquad /1.27a/$$

In a similar way it can be proved that

$$A^{0,j} = \left(A^{0,1}\right)^j \quad \text{for} \quad j = 1,2,\ldots \qquad /1.27b/$$

By induction on $/i,j/$ we shall show that

$$A^{1,0}A^{i-1,j} + A^{0,1}A^{i,j-1} = A^{i-1,j}A^{1,0} + A^{i,j-1}A^{0,1} \qquad /1.27c/$$

For $/i,j/ = /0,0/$ equation /1.25/ yields

$$A^{1,0}A^{-1,0} + A^{0,1}A^{0,-1} = A^{-1,0}A^{1,0} + A^{0,-1}A^{0,1}$$

and for $/i,j/ = /1,0/$, $/i,j/ = /0,1/$

$$A^{1,0}A^{0,0} + A^{0,1}A^{1,-1} = A^{0,0}A^{1,0} + A^{1,-1}A^{0,1}$$

$$A^{1,0}A^{-1,1} + A^{0,1}A^{0,0} = A^{-1,1}A^{1,0} + A^{0,0}A^{0,1}$$

Thus /1.27c/ is true for $/i,j/ = /0,0/$, $/i,j/ = /1,0/$ and $/i,j/ = /0,1/$. In a similar way it can be proven true for $/i,j/ = /i_o,1/$ and $/i,j/ = /1,j_o/$; $i_o, j_o > 1$. Assuming that the hypothesis /1.27c/ is true for all $/k,l/$ such that $/0,0/ < /k,l/ < /i,j/$ it will be shown that it is valid for $/i,j/$.

$$A^{1,0}\left(A^{i-2,j}A^{1,0} + A^{i-1,j-1}A^{0,1}\right) + A^{0,1}\left(A^{i-1,j-1}A^{1,0} + A^{i,j-2}A^{0,1}\right) =$$
$$= \left(A^{1,0}A^{i-2,j} + A^{0,1}A^{i-1,j-1}\right)A^{1,0} + \left(A^{1,0}A^{i-1,j-1} + A^{0,1}A^{i,j-2}\right)A^{0,1} =$$
$$= A^{i-1,j}A^{1,0} + A^{i,j-1}A^{0,1}.$$

This completes the proof of /1.27c/.

Theorem 1.1 [13]

A solution to the equation /1.1/ with boundary conditions /1.1a/ is given by

$$\begin{bmatrix} x^h/i,j/ \\ x^v/i,j/ \end{bmatrix} = \sum_{k_1=0}^{i} A^{i-k_1,j} \begin{bmatrix} 0 \\ x^v/k_1,0/ \end{bmatrix} + \sum_{k_2=0}^{j} A^{i,j-k_2} \begin{bmatrix} x^h/0,k_2/ \\ 0 \end{bmatrix} +$$

$$+ \sum_{/0,0/\leq/k_1,k_2/</i,j/} \left[A^{i-k_1-1,j-k_2} B^{1,0} + A^{i-k_1,j-k_2-1} B^{0,1} \right] u/k_1,k_2/ \qquad /1.28/$$

where

$$B^{1,0} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad B^{0,1} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \qquad /1.25a/$$

Proof: The proof is accomplished using induction on /i,j/.
For /i,j/ = /0,0/ equation /1.1/ yields

$$\begin{bmatrix} x^h/1,0/ \\ x^v/0,1/ \end{bmatrix} = A \begin{bmatrix} x^h/0,0/ \\ x^v/0,0/ \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u/0,0/$$

The same result follows from /1.28/. Thus the hypothesis is true for /i,j/=/0,0/. It is a matter of bookkeeping to show that the hypothesis holds true also for /i,j/ = /i_0,0/, /0,j_0/; $i_0, j_0 > 0$. Assuming that the hypothesis is true for all /k_1,k_2/ such that /0,0/ $<$ /k_1,k_2/ $<$ /i,j/, it will be shown that the hypothesis is valid for /i,j/.
From /1.1/ it follows that

$$\begin{bmatrix} x^h/i,j/ \\ x^v/i,j/ \end{bmatrix} = A^{1,0} \begin{bmatrix} x^h/i-1,j/ \\ x^v/i-1,j/ \end{bmatrix} + A^{0,1} \begin{bmatrix} x^h/i,j-1/ \\ x^v/i,j-1/ \end{bmatrix} + B^{1,0} u/i-1,j/ +$$

$$+ B^{0,1} u/i,j-1/ \qquad /1.29/$$

Substituting the expressions for

$$\begin{bmatrix} x^h/i-1,j/ \\ x^v/i-1,j/ \end{bmatrix}, \begin{bmatrix} x^h/i,j-1/ \\ x^v/i,j-1/ \end{bmatrix} \qquad \text{/which follow from /1.28//}$$

into /1.29/ we obtain

$$\begin{bmatrix} x^h/i,j/ \\ x^v/i,j/ \end{bmatrix} = A^{1,0} \left\{ \sum_{k_1=0}^{i-1} A^{i-k_1-1,j} \begin{bmatrix} 0 \\ x^v/k_1,0/ \end{bmatrix} + \sum_{k_2=0}^{j} A^{i-1,j-k_2} \begin{bmatrix} x^h/0,k_2/ \\ 0 \end{bmatrix} + \right.$$

$$\left. + \sum_{/0,0/\leq/k_1,k_2/</i-1,j/} \left[A^{i-k_1-2,j-k_2} B^{1,0} + A^{i-k_1-1,j-k_2-1} B^{0,1} \right] u/k_1,k_2/ \right\} +$$