An Introduction to Nonharmonic Fourier Series

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PREFACE

The theory of nonharmonic Fourier series is concerned with the completeness and expansion properties of sets of complex exponentials $\{e^{i\lambda_n t}\}$ in $L^p[-\pi,\pi]$. Its origins, which are classical in spirit, lie in the celebrated works of Paley and Wiener [1934] and Levinson [1940]. In recent years, in response to the development of functional analysis and, in particular, to the growing interest in bases in Banach spaces, research in the area has flourished. New approaches to old problems have led to important advances in the theory.

This book is an account of both the classical and the modern theories. Its underlying theme is the elegant interplay among the various parts of analysis. The catalyst in the present case is the Fourier transform, through which the classical Banach spaces are mapped into spaces of entire functions. In this way, problems in one domain can be examined via their transform image in the other.

The book is designed primarily for the graduate student or mathematician who is approaching the subject for the first time. Its aim as such is to provide a unified and self-contained introduction to a multifaceted field, not an exhaustive account of all that is known. Accordingly, the first half of the book presents an elementary introduction to the theory of bases in Banach spaces and the theory of entire functions of exponential type. At the same time, an extensive set of notes touches on more advanced topics, indicates directions in which the theory can be extended, and should prove useful to both specialists and nonspecialists alike. Much of the material appears in book form for the first time.

The only prerequisites are a working knowledge of real and complex analysis, together with the elements of functional analysis. By that I mean roughly what is contained in Rudin [1966]. On occasion, when more advanced tools of analysis are required, appropriate references are given. Apart from this, the work is essentially self-contained, and it can serve as a textbook for a course at the second- or third-year graduate level.

The problems, which are of varying degrees of difficulty, are an integral part of the text. Some are routine applications of the theory, while others are important ancillary results—these are usually accompanied by an indication of the solution and an appropriate reference to the literature.

A word about notation: Theorem 2.3 refers to Theorem 3 of Chapter 2. The labeling of all other results is self-explanatory.

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ROBERT M. YOUNG

CONTENTS

1 *BASES IN BANACH SPACES*

1 Schauder Bases

Let X be an infinite-dimensional Banach space over the field of real or complex numbers. When viewed as a vector space, X is known to possess a *Hamel basis*—a linearly independent subset of X that spans the entire space. Unfortunately, such bases cannot in general be constructed, their very existence depending on the axiom of choice, and their usefulness is therefore severely limited. Of far greater importance and applicability in analysis is the notion of a basis first introduced by Schauder [1927].

Definition. *A sequence of vectors $\{x_1, x_2, x_3, \ldots\}$ in an infinite-dimensional Banach space X is said to be a* **Schauder basis** *for X if to each vector x in the space there corresponds a unique sequence of scalars $\{c_1, c_2, c_3, \ldots\}$ such that*

$$x = \sum_{n=1}^{\infty} c_n x_n.$$

The convergence of the series is understood to be with respect to the strong (norm) topology of X; in other words,

$$\left\| x - \sum_{i=1}^{n} c_i x_i \right\| \to 0 \qquad \text{as} \quad n \to \infty.$$

Henceforth, the term *basis* for an infinite-dimensional Banach space will always mean a Schauder basis.

Example. The Banach space l^p $(1 \leqq p < \infty)$ consists, by definition, of all infinite sequences of scalars $c = \{c_1, c_2, c_3, \ldots\}$ such that $\|c\|_p = (\sum_{n=1}^{\infty} |c_n|^p)^{1/p} < \infty$. The vector operations are coordinatewise. In each of these spaces, the "natural basis" $\{e_1, e_2, e_3, \ldots\}$, where

$$e_n = (0, 0, \ldots, 0, 1, 0, \ldots),$$

and the 1 appears in the nth position, is easily seen to be a Schauder basis. If $c = \{c_n\}$ is in l^p, then the obvious expansion $c = \sum_{n=1}^{\infty} c_n e_n$ is valid.

It is clear that a Banach space with a basis must be *separable*. Reason: If $\{x_n\}$ is a basis for X, then the set of all finite linear combinations $\sum c_n x_n$, where the c_n are *rational* scalars, is countable and dense in X. It follows, for example, that since l^∞ is not separable, it cannot possess a basis.

The "basis problem"—whether or not every separable Banach space has a basis—was raised by Banach [1932] and remained until recently one of the outstanding unsolved problems of functional analysis. The question was finally settled by Per Enflo [1973], who constructed an example of a separable Banach space having no basis. The negative answer to the basis problem is perhaps surprising in light of the fact that bases are now known for almost all the familiar examples of infinite-dimensional separable Banach spaces.

Problems

1. Prove that every vector space has a Hamel basis.
2. Prove that every Hamel basis for a given vector space has the same number of elements. This number is called the (linear) **dimension** of the space.
3. Show that a Hamel basis for an infinite-dimensional Banach space is uncountable.
4. Show that the dimension of l^∞ is equal to c. (*Hint:* Show that the set $\{(1, r, r^2, \ldots) : 0 < r < 1\}$ is linearly independent.)
5. Let X be an infinite-dimensional Banach space.
   (a) Prove that $\dim X \geqq c$. (*Hint:* Show that there is a vector space isomorphism between l^∞ and a subspace of X.)
   (b) Prove that if X is separable, then $\dim X = c$.
6. The Banach space c_0 consists of all infinite sequences of scalars which converge to zero (with the l^∞ norm). Show that the natural basis is a Schauder basis for c_0.
7. Exhibit a Schauder basis for the Banach space c consisting of all convergent sequences of scalars (with the l^∞ norm).
8. An infinite series $\sum x_n$ in a Banach space X is said to be **unconditionally convergent** if every arrangement of its terms converges to the same element. It is said to be **absolutely convergent** if the series $\sum \|x_n\|$ is

convergent. Show that every absolutely convergent series in X is unconditionally convergent. What about the converse?

9. A basis $\{x_n\}$ for a Banach space X is said to be **unconditional (absolute)** if every convergent series of the form $\sum c_n x_n$ is unconditionally (absolutely) convergent.
   (a) Show that the natural basis is unconditional for the spaces l^p, $1 \leqq p < \infty$, and c_0. Show also that it is absolute for l^p only when $p = 1$. Is it absolute for c_0?
   (b) Show that the sequence of vectors
   $$(1, 0, 0, 0, \ldots), \quad (1, 1, 0, 0, \ldots), \quad (1, 1, 1, 0, \ldots), \quad \ldots$$
   forms a basis for c_0 which is not unconditional.

2 Schauder's Basis for $C[a, b]$

One of the most important and widely studied classical Banach spaces is $C[a, b]$, the space of all continuous functions on the closed finite interval $[a, b]$, together with the norm

$$\|f\| = \max|f(x)|.$$

The celebrated Weierstrass approximation theorem asserts that the polynomials are *dense* in $C[a, b]$: if f is continuous on $[a, b]$, then for every positive number ε there is a polynomial P such that the inequality

$$|f(x) - P(x)| < \varepsilon$$

holds throughout the interval $[a, b]$.

For a given continuous function, a sequence of approximating polynomials can even be given explicitly. The most elegant representation is due to Bernstein. Let us suppose, for simplicity, that f is continuous on the interval $[0, 1]$. Then the nth *Bernstein polynomial* for f is

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1 - x)^{n-k}, \qquad n = 1, 2, 3, \ldots.$$

As is well known,

$$f(x) = \lim_{n \to \infty} B_n(x)$$

uniformly on $[0, 1]$ (see Akhiezer [1956, p. 30]).

Since every polynomial can be uniformly approximated on a closed interval by a polynomial with *rational* coefficients, the preceding remarks show that the space $C[a, b]$ is separable; in fact, it has a basis.

Theorem 1 (Schauder). *The space $C[a, b]$ possesses a basis.*

Proof. We are going to construct a basis for $C[a, b]$ consisting of piecewise-linear functions f_n $(n = 0, 1, 2, \ldots)$. This means that to each function f in the space there will correspond a unique sequence of scalars $\{c_n\}$ such that

$$f(x) = \sum_{n=0}^{\infty} c_n f_n(x)$$

uniformly on $[a, b]$.

Let $\{x_0, x_1, x_2, \ldots\}$ be a countable dense subset of $[a, b]$ with $x_0 = a$ and $x_1 = b$. Set

$$f_0(x) = 1 \qquad \text{and} \qquad f_1(x) = \frac{x - a}{b - a}.$$

When $n \geqq 2$, the set of points $\{x_0, x_1, \ldots, x_{n-1}\}$ partitions $[a, b]$ into disjoint open intervals, one of which contains x_n; call it I. Define

$$f_n(x) = \begin{cases} 0 & \text{if} \quad x \notin I \\ 1 & \text{if} \quad x = x_n \\ \text{linear} & \text{elsewhere} \end{cases}$$

for $n = 2, 3, 4, \ldots$. The sequence $\{f_0, f_1, f_2, \ldots\}$ will be the required basis.

For each function f in $C[a, b]$ and each positive integer n, we denote by $L_n f$ the polygonal function that agrees with f at each of the points $x_0, x_1, \ldots, x_n$; we denote by $L_0 f$ the function whose constant value is $f(x_0)$. Since f is uniformly continuous on $[a, b]$, a simple continuity argument shows that

$$L_n f \to f \qquad \text{uniformly on} \quad [a, b].$$

Therefore, we can write

$$f = L_0 f + \sum_{n=1}^{\infty} (L_n f - L_{n-1} f).$$

We are going to show that there are scalars $c_1, c_2, c_3, \ldots$ such that

$$L_n f - L_{n-1} f = c_n f_n \qquad (n = 1, 2, 3, \ldots).$$

For this purpose, we shall define a sequence of functions $\{g_0, g_1, g_2, \ldots\}$ recursively by the equations

$$g_0 = f(x_0)f_0 \quad \text{and} \quad g_n = g_{n-1} + (f - g_{n-1})(x_n)f_n, \qquad n = 1, 2, 3, \ldots.$$

The claim is that $g_n = L_n f$, whence

$$c_n = (f - L_{n-1}f)(x_n).$$

Since g_n is a polygonal function whose only possible corners are at the points $x_0, x_1, \ldots, x_n$, it is sufficient to show that g_n agrees with f at each of these points. This is trivial for $n = 0$, and we proceed by induction. Since $f_n(x_n) = 1$, it follows that $g_n(x_n) = f(x_n)$; if $i < n$, then $f_n(x_i) = 0$, and it follows from the definition of g_n, together with the induction hypothesis, that

$$g_n(x_i) = g_{n-1}(x_i) = f(x_i).$$

This establishes the claim.

Accordingly, every function $f \in C[a, b]$ has at least one representation of the form

$$f = \sum_{n=0}^{\infty} c_n f_n,$$

and we have only to show that this representation is unique. Suppose then that some function g has two different representations, say $\sum_{n=0}^{\infty} a_n f_n$ and $\sum_{n=0}^{\infty} b_n f_n$. If N is the smallest value of n for which $a_n \neq b_n$, then

$$\sum_{n=N}^{\infty} a_n f_n(x) = \sum_{n=N}^{\infty} b_n f_n(x)$$

for every x. Choose $x = x_N$. Since $f_n(x_N) = 0$ whenever $n > N$, it follows that $a_N = b_N$. But this contradicts the choice of N, and hence $a_n = b_n$ for every n. ∎

Problems

1. Give a probabilistic interpretation of the Bernstein polynomials (*see* Feller [1966, Chap. VII]).

2. Prove that the space $C[a, b]$ is separable by showing that every continuous function on $[a, b]$ can be uniformly approximated by polynomials with *rational* coefficients.
3. Let f be a continuous function on $(-\infty, \infty)$. Prove that if there is a sequence of polynomials $\{P_1, P_2, P_3, \ldots\}$ such that $P_n \to f$ uniformly on $(-\infty, \infty)$, then f must itself be a polynomial.
4. Let f be a continuous function on $[a, b]$. Show that there is a sequence of polynomials $\{P_1, P_2, P_3, \ldots\}$ such that $f = \sum_{n=1}^{\infty} P_n$ and the series converges absolutely and uniformly on $[a, b]$.

3 Orthonormal Bases in Hilbert Space

In a separable Hilbert space†, a distinguished role is played by those Schauder bases that are *orthonormal*—the basis vectors are mutually perpendicular and each has unit length. An equivalent characterization of such bases is that they are *complete* orthonormal sequences. (Recall that a sequence of vectors $\{f_1, f_2, f_3, \ldots\}$ in a Hilbert space is said to be complete if the zero vector alone is perpendicular to every f_n.) It follows readily from this characterization that every separable Hilbert space has an orthonormal basis.

The most important property of an orthonormal basis (as opposed to any other basis) is the simplicity of all basis expansions. If $\{e_1, e_2, e_3, \ldots\}$ is an orthonormal basis for a Hilbert space H, then for every element $f \in H$ we have the *Fourier expansion*

$$f = \sum_{n=1}^{\infty} (f, e_n)e_n.$$

The inner product (f, e_n) is called the nth *Fourier coefficient* of f (relative to $\{e_n\}$). When the Pythagorean formula is applied to this series, the result is **Parseval's identity:**

$$\|f\|^2 = \sum_{n=1}^{\infty} |(f, e_n)|^2.$$

The validity of Parseval's identity for every vector in the space is both necessary and sufficient for an orthonormal sequence to be a basis.

† All Hilbert spaces are assumed to be infinite-dimensional.

Since the linear transformation

$$f \to \{(f, e_n)\}$$

from H into l^2 preserves norms, it must also preserve inner products. Thus

$$(f, g) = \sum_{n=1}^{\infty} (f, e_n)\overline{(g, e_n)}$$

for every pair of vectors f and g; this is the *generalized* Parseval identity.

Even if an orthonormal sequence $\{e_n\}$ is incomplete, **Bessel's inequality** is always valid:

$$\sum_{n=1}^{\infty} |(f, e_n)|^2 \leqq \|f\|^2$$

whenever $f \in H$. This shows, in particular, that the Fourier coefficients of each element of H form a square-summable sequence. The **Riesz–Fischer theorem** shows, conversely, that every square-summable sequence is obtained in this way: if $\sum_{n=1}^{\infty} |c_n|^2 < \infty$, then there exists an element f in H for which

$$(f, e_n) = c_n, \qquad n = 1, 2, 3, \ldots.$$

The proof is trivial: simply choose $f = \sum_{n=1}^{\infty} c_n e_n$. We conclude that if $\{e_n\}$ is a *complete* orthonormal sequence in H, then the correspondence $f \to \{(f, e_n)\}$ between H and l^2 is a Hilbert space isomorphism. It follows that from a geometric point of view, all separable Hilbert spaces are "indistinguishable", that is to say, isomorphic.

Example 1. In l^2 the "natural basis" is orthonormal.

Example 2. In $L^2[-\pi, \pi]$, with the inner product

$$(f, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\overline{g(t)}\, dt,$$

the complex trigonometric system $\{e^{int}\}_{-\infty}^{\infty}$ constitutes an orthonormal basis. That the system is orthonormal is obvious; we prove that it is complete.

Theorem 2. *The trigonometric system is complete in $L^2[-\pi, \pi]$.*

Proof. The proof will establish even more. Suppose that

$$\int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt = 0$$

for some *integrable* function f defined on $[-\pi, \pi]$ and $n = 0, \pm 1, \pm 2, \ldots$. It is to be shown that $f = 0$ a.e. Set

$$g(t) = \int_{-\pi}^{t} f(u)\, du$$

for $t \in [-\pi, \pi]$. Integration by parts shows that

$$\int_{-\pi}^{\pi} (g(t) - c)\, e^{-int}\, dt = 0$$

for every constant c and $n = \pm 1, \pm 2, \pm 3, \ldots$. Choose c so that this holds for $n = 0$ also, and put

$$F(t) = g(t) - c.$$

Then F is continuous on $[-\pi, \pi]$ and $F(\pi) = F(-\pi)$. Weierstrass's theorem on approximation by trigonometric polynomials guarantees that for each $\varepsilon > 0$ there is a finite trigonometric sum

$$T(t) = \sum_{k=-n}^{n} c_k e^{ikt}$$

such that

$$|F(t) - T(t)| < \varepsilon \qquad \text{whenever} \quad |t| \leqq \pi.$$

It follows that

$$\|F\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(t)|^2\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{F(t)}(F(t) - T(t))\, dt$$

$$\leqq \frac{\varepsilon}{2\pi} \int_{-\pi}^{\pi} |F(t)|\, dt \leqq \varepsilon \|F\|,$$

so that

$$\|F\| \leqq \varepsilon.$$

Since ε was arbitrary, $F = 0$, so that $g = c$ and $f = 0$ a.e. ∎

Consequently, every function f in $L^2[-\pi, \pi]$ has a unique Fourier series expansion

$$f(t) = \sum_{-\infty}^{\infty} \hat{f}(n)e^{int}$$

(*in the mean*†). Here $\hat{f}(n)$ denotes the nth Fourier coefficient of f relative to $\{e^{int}\}$, i.e.,

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

By Parseval's formula,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2\,dt = \sum_{-\infty}^{\infty} |\hat{f}(n)|^2.$$

The mapping $f \to \{\hat{f}(n)\}$ is a Hilbert space isomorphism between $L^2[-\pi, \pi]$ and l^2.

There is a simple but useful extension of Parseval's identity that is worth mentioning. If $f \in L^2[-\pi, \pi]$, let $\hat{f}$ be the **Fourier transform** of f:

$$\hat{f}(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ixt}\,dt \qquad (-\infty < x < \infty).$$

Proposition 1. *For every function $f \in L^2[-\pi, \pi]$ and every real number A,*

$$\sum_{-\infty}^{\infty} |\hat{f}(n + A)|^2 = \|f\|^2.$$

Proof. Put $g(t) = f(t)e^{-iAt}$. Then

$$\hat{f}(n + A) = \hat{g}(n)$$

† Pointwise convergence is of course much harder. A deep result of Carleson [1966] says that the Fourier series of an L^2 function converges (to the function) pointwise almost everywhere.

for every integer n. Since A is real, $\|f\| = \|g\|$, and the result follows from Parseval's identity applied to g. ∎

As an illustration, let us choose f to be the constant function 1. A simple calculation shows that

$$\hat{f}(x) = \frac{\sin \pi x}{\pi x}$$

for all real x. Setting $A = t/\pi$, where t is real and not an integral multiple of π, we obtain the important identity

$$\frac{1}{\sin^2 t} = \sum_{-\infty}^{\infty} \frac{1}{(n\pi + t)^2}.$$

Example 3. The space $\mathbf{H}^2$ (named after Hardy) consists of all functions f analytic in the open unit disk (in the complex plane) whose Taylor coefficients are square-summable, i.e.,

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \qquad \text{with} \quad \sum_{n=0}^{\infty} |c_n|^2 < \infty.$$

The inner product of two functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ in $\mathbf{H}^2$ is, by definition,

$$(f, g) = \sum_{n=0}^{\infty} a_n \bar{b}_n.$$

It is clear that $\mathbf{H}^2$ can be identified with the (closed) subspace of $L^2[-\pi, \pi]$ spanned by the functions e^{int} with $n \geqq 0$.

Let $e_n(z) = z^n$ for $|z| < 1$ $(n = 0, 1, 2, \ldots)$; then the e_n's form an orthonormal basis for $\mathbf{H}^2$. The natural mapping

$$(c_0, c_1, c_2, \ldots) \to \sum_{n=0}^{\infty} c_n z^n$$

between l^2 and $\mathbf{H}^2$ is a Hilbert space isomorphism.

Example 4. The space $\mathbf{A}^2$ consists of all functions f that are analytic in