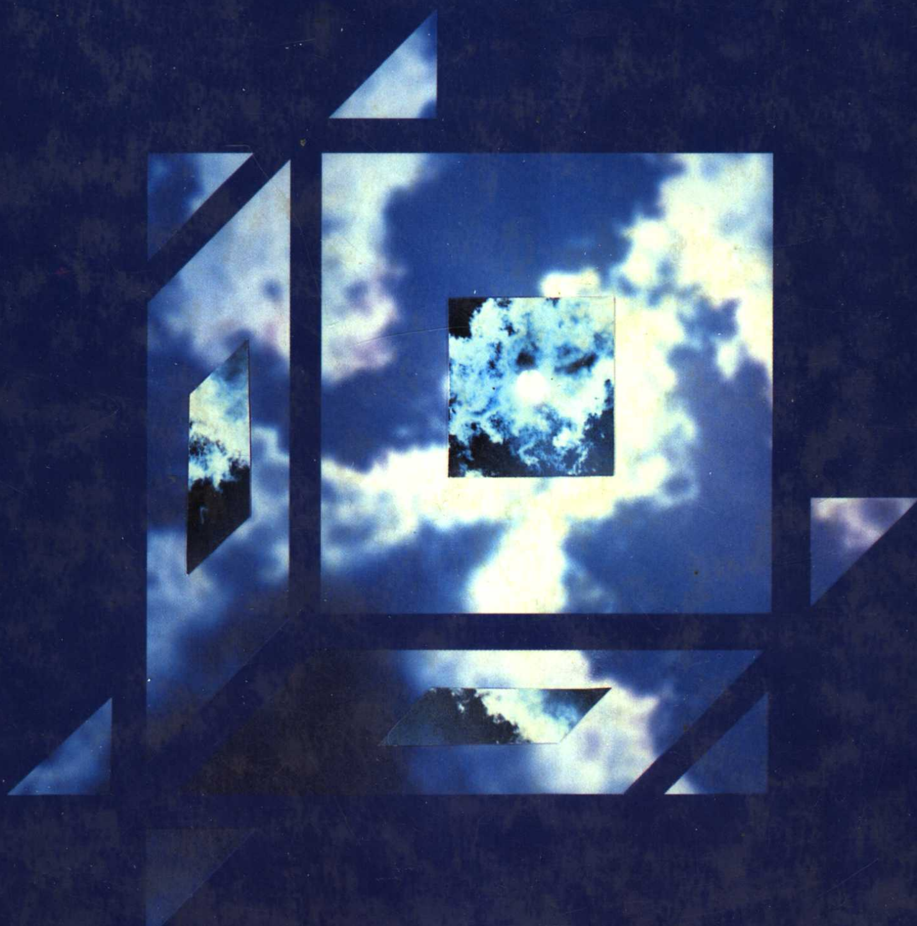
Elementary Linear Algebra
with Applications

W. Keith Nicholson



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University of Calgary

Prindle, Weber & Schmidt
Boston

Published simultaneously in Canada by Wadsworth Publishers of Canada, Ltd., Toronto

PWS PUBLISHERS

Prindle, Weber & Schmidt • Duxbury Press • PWS Engineering • Breton Publishers •
20 Park Plaza • Boston, Massachusetts 02116

PWS Publishers is a division of Wadsworth, Inc.

Library of Congress Cataloging in Publication Data

Nicholson, W. Keith.
Elementary linear algebra, with applications.

Includes index.
1. Algebras, Linear. I. Title.
QA184.N53 1986 512'.5 85-25769
ISBN 0-87150-902-4

ISBN 0-87150-902-4

Printed in the United States of America

87 88 89 90 — 10 9 8 7 6 5 4

Sponsoring Editor Harry Campbell/David Geggis
Production Coordinator Susan Graham
Manuscript Editor Connie Day
Interior Design Susan Graham
Cover Design Susan Graham
Cover Photo: © 1985 Sanjay Kothari
Interior Illustration: Deborah Schneck
Typesetting: Weimer Typesetting
Cover Printing: New England Book Components
Printing and Binding: The Maple-Vail Book Manufacturing Group

Preface

Elementary Linear Algebra with Applications is a basic introduction to the results and techniques of linear algebra for students with only a good knowledge of high school algebra. An acquaintance with linear algebra has long been a requirement for students of science, mathematics, and computing science; it is now commonly required in other areas such as management and economics. As a result, beginning linear algebra courses often rival calculus in enrollment and include many students who are not particularly mathematically inclined. Many of the present books on the subject, therefore, are so computationally oriented that the mathematics is all but ignored. This orientation makes for dull teaching and has the effect of turning potential mathematics majors away from the subject. This book aims at achieving a balance between computational skills, theory, and applications of linear algebra while keeping the level suitable for beginning students.

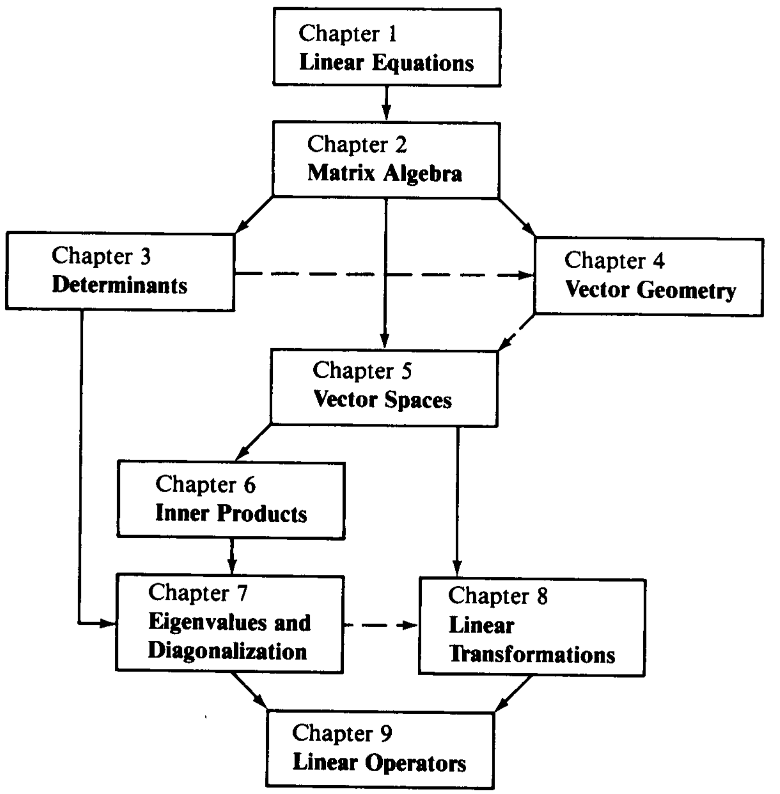
My goal in writing this book can be summed up in a quotation from Albert Einstein: "Everything should be made as simple as possible, but not simpler." Making this material accessible to students does not necessarily mean lowering the level. The following features help to make this text mathematically interesting and yet accessible to the vast majority of nonmathematical students who take the course:

- Over 275 solved examples (not including applications) that are keyed to the exercises;

- Presentation of techniques in examples, with an emphasis on concrete computations, to introduce methods later used in proofs (many of which are optional);
- Choice of following a rigorous treatment using optional proofs, or a methods approach using the examples but omitting many of the proofs;
- Exercise sets beginning with routine problems and proceeding to more theoretical exercises, with answers to even-numbered computational exercises at the back of the book;
- Wide variety of applications at the end of each chapter where linear algebra gives new insight, rather than merely playing a descriptive role.

The tables of contents of all linear algebra texts are much the same because wide agreement exists on the topics that should be included. *Elementary Linear Algebra with Applications* is no exception, although I have included some special features. First, the vector geometry (Chapter 4) can be omitted since many students get this material elsewhere. Second, diagonalization of matrices (Chapter 7) may be covered

CHAPTER DEPENDENCIES



— → This indicates that some reference is made but the chapter need not be covered.

prior to linear transformations, thus opening up the possibility of a one-semester methods course (Chapters 1, 2, 3, 4, 5, 7). Finally, although most instructors will not have much extra time, applications sections are included because they are useful pedagogically and their location in the same chapter as the relevant linear algebra will encourage the better students to browse.

Additional features include the following:

- Appendix on linear programming (requires only Chapter 1). This is a popular option and is a natural extension of Gauss-Jordan elimination.
- Appendices on complex numbers and induction. Complex numbers are used in the text to prove that eigenvalues of symmetric matrices are real.
- Emphasis on the algorithmic nature of several of the techniques.
- Flexibility in the ordering of chapters. In particular, Chapter 4 can be omitted and eigenvalues and diagonalization can be treated before linear transformations.
- Solutions manual available to the instructor.

Acknowledgments

I would like to record my appreciation to the following reviewers for their useful comments and suggestions: G. D. Allen, Texas A & M University; R. F. V. Andersen, University of British Columbia; Graham A. Chambers, University of Alberta; Henry P. Decell Jr., University of Houston; Garret J. Etgen, University of Houston; Ivan Gombos, Dawson College; Eugene W. Johnson, University of Iowa; B. J. Kirby, Queen's University; Stanley O. Kochman, York University; Peter Lancaster, University of Calgary; Joe Repka, University of Toronto; L. G. Roberts, University of British Columbia; A. H. Low, University of New South Wales; Bostwick F. Wyman, Ohio State University.

I would like to extend special thanks to my editor Harry Campbell of Wadsworth, Canada Ltd. for his enthusiastic encouragement throughout the project and to Ron Munro for his help during the initial stages. Thanks are also due to the production staff at Prindle, Weber & Schmidt, to Connie Day, and particularly to Susan Graham. I am most grateful to Joanne Longworth and Gisele Vezina who typed the manuscript, to Jason Brown who helped with the exercises, and to Jason Nicholson who helped with the proofreading. Finally, I want to thank my wife, Kathleen, for her unfailing support.

W. Keith Nicholson

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CHAPTER 1

Systems of Linear Equations

SECTION 1.1 INTRODUCTION

One of the great historical motivations for the development of mathematics has been to find a way to analyze and solve practical problems. This need prompted the use of numbers and of geometry, certainly two of the most basic mathematical systems. Certain types of problems led naturally to systems of linear equations that, when solved, gave useful practical information. Linear algebra arose from attempts to find systematic methods for solving these systems, so it is natural to begin this book by studying linear equations.

To see how systems of linear equations arise in practical situations, consider the following example.

EXAMPLE 1 A charity wishes to endow a fund that will supply \$50,000 per year for cancer research. The charity has \$480,000 and, for reasons of security, wants to invest it with two banks, one paying 10% per year and the other paying 11%. The question is: How much should be invested in each bank?

SOLUTION If x is the amount invested at 10% and y is the amount invested at 11%, then $x + y = 480{,}000$ and the yearly interest is $\frac{10}{100}x + \frac{11}{100}y$. Hence x and y must satisfy the conditions

$$\begin{aligned} x + y &= 480{,}000 \\ \frac{10}{100}x + \frac{11}{100}y &= 50{,}000 \end{aligned}$$

If the first of these equations is multiplied by 10 and the second by 100, the resulting equations are

$$10x + 10y = 4{,}800{,}000$$
$$10x + 11y = 5{,}000{,}000$$

Subtracting the first from the second gives $y = 200{,}000$, and therefore $x = 280{,}000$. □

This example typifies the way in which linear equations arise, and even the method of solution is typical of the methods used in more general situations. Other examples will differ from this in two ways: More than two variables will usually be needed, and more than two equations will be found that these variables must satisfy.

If a, b, and c are real numbers, the graph of an equation of the form

$$ax + by = c$$

is a straight line (provided that a and b are not both zero). Accordingly, such an equation is called a linear equation in the variables x and y. When only two or three variables are present, they are usually called x, y, and z. However, it is often convenient to write the variables as $x_1, x_2, \ldots, x_n$, particularly when more than three variables are involved. An equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

is called a **linear equation** in the n variables $x_1, x_2, \ldots, x_n$. Here $a_1, a_2, \ldots, a_n$ denote real numbers (called the **coefficients** of $x_1, x_2, \ldots, x_n$, respectively) and b is also a number (called the **constant term** of the equation). Hence

$$2x_1 - 3x_2 + 5x_3 = 7 \qquad \text{and}$$
$$x_1 + x_2 + x_3 + x_4 = 0$$

are both linear equations. Note that each variable in a linear equation occurs to the first power only, so the following are *not* linear equations.

$$x_1^2 + 3x_2 - 2x_3 = 5$$
$$x_1 + x_1x_2 + 2x_3 = 1$$
$$\sqrt{x_1} + x_2 - x_3 = 0$$
$$x_1 + x_2 - 3x^3 = 1$$

Consider the single linear equation $3x_1 + 2x_2 - x_3 = 1$. A solution to this equation is a triple of numbers s_1, s_2, s_3, denoted $\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$, with the property that the equation is satisfied when the substitutions $x_1 = s_1$, $x_2 = s_2$, and $x_3 = s_3$ are made. For example, $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}$ are solutions to this equation but $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is not. Note that the *order* of the numbers s_1, s_2, s_3 in a solution is important. For example, $\begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$ is a solution to the foregoing equation, whereas $\begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}$ is not a solution (even though it consists of the same numbers in a different order). For this reason the notation $\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$ denotes an **ordered triple** of numbers, and two such triples are regarded as equal only when corresponding entries are equal.

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \quad \text{means} \quad s_1 = t_1,\ s_2 = t_2,\ s_3 = t_3$$

More generally, an **ordered *n*-tuple** $\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$ is an ordered sequence of n numbers $s_1, s_2, \ldots, s_n$ (called the **entries** of the n-tuple). Two such n-tuples are defined to be equal only when corresponding entries are equal.

$$\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix} \quad \text{means} \quad s_1 = t_1,\ s_2 = t_2,\ \ldots,\ s_n = t_n$$

For example, $\begin{bmatrix} 2 \\ 1 \\ 3 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$ whereas $\begin{bmatrix} 1 \\ 2 \\ x \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \\ 0 \end{bmatrix}$ if and only if $x = 5$. The ordered 3-tuples are, of course, just the ordered triples discussed above, and ordered 2-tuples are called **ordered pairs.** Incidentally, ordered n-tuples can also be written as rows.

Suppose a linear equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

is given. An ordered n-tuple $\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$ is called a **solution** to the equation if $a_1s_1 + a_2s_2 + \cdots + a_ns_n = b$—that is, if the equation is satisfied when the substitutions $x_1 = s_1$, $x_2 = s_2$, $\ldots$, $x_n = s_n$ are made. A finite collection of linear equations in the variables $x_1, x_2, \ldots, x_n$ is called a **system of linear equations** in these variables. An ordered n-tuple is called a **solution to the system** of equations if it is a solution to every equation in the system. The system of two equations in two variables that arose in Example 1 had a unique solution, $\begin{bmatrix} 280000 \\ 200000 \end{bmatrix}$. However, other possibilities exist. A system may have no solution at all, or it may have an infinite family of solutions. For example, the system $x + y = 2$, $x + y = 3$ has no solution, whereas Example 2 exhibits a system with infinitely many solutions. The aim in general is to **solve** the system of linear equations—that is, to find *all* solutions to the system. This chapter is devoted primarily to developing a systematic method for doing this.

EXAMPLE 2 Show that $\begin{bmatrix} t - s - 1 \\ t + s + 1 \\ s \\ t \end{bmatrix}$ is a solution to the system

$$x_1 - 2x_2 + 3x_3 + x_4 = -3$$
$$2x_1 - x_2 + 3x_3 - x_4 = -3$$

for any values of s and t.

SOLUTION Simply substitute $x_1 = t - s - 1$, $x_2 = t + s + 1$, $x_3 = s$, and $x_4 = t$ in each equation.

$$x_1 - 2x_2 + 3x_3 + x_4 = (t - s - 1) - 2(t + s + 1) + 3s + t = -3$$
$$2x_1 - x_2 + 3x_3 - x_4 = 2(t - s - 1) - (t + s + 1) + 3s - t = -3$$

Because both equations are satisfied, it is a solution for all s and t. □

The solutions given in Example 2 can be written as follows:

$$\begin{aligned} x_1 &= t - s - 1 \\ x_2 &= t + s + 1 \\ x_3 &= s \\ x_4 &= t \end{aligned} \qquad (s \text{ and } t \text{ arbitrary})$$

This means that, for any choice of s and t, the values of x_1, x_2, x_3, and x_4 given by these equations will satisfy the equations. The quantities s and t are called **parameters,** and this set of solutions, described in this way, is said to be given in **parametric form.** It turns out that solutions to systems of linear equations quite often appear in this form and that such descriptions arise naturally. The following examples show how this comes about in the simplest systems where only one equation is present.

EXAMPLE 3 Describe all solutions to $3x - y = 4$ in parametric form.

SOLUTION The equation can be written in the form

$$y = 3x - 4$$

Hence, if t denotes *any* number at all, we can quite arbitrarily set $x = t$ and then obtain $y = 3t - 4$. This is clearly a solution to our equation for any value of t. On the other hand, *every* solution to $3x - y = 4$ arises in this way (t is just the value of x). Hence the set of *all* solutions can be described parametrically as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 3t - 4 \end{bmatrix} \qquad t \text{ arbitrary}$$

Note that there are *infinitely* many distinct solutions, one for each choice of the parameter t.

It is important to realize that the solutions to $3x - y = 4$ can be given in parametric form in several ways. We found the foregoing solution by observing that $y = 3x - 4$ and then choosing $x = t$, t a parameter. However, we could have found x in terms of y:

$$x = \frac{1}{3}(y + 4)$$

and then chosen $y = s$ (s a parameter). Hence the solutions are

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3}(s + 4) \\ s \end{bmatrix} \qquad s \text{ arbitrary}$$

This is also a correct parametric representation of the solutions to $3x - y = 4$. In fact, the parameters are related by $s = 3t - 4$ (or $t = \frac{1}{3}(s + 4)$). □

EXAMPLE 4 Describe all solutions to $3x - y + 2z = 6$ in parametric form.

SOLUTION Solving the equation for y in terms of x and z, we get $y = 3x + 2z - 6$. Then x and z can be arbitrarily chosen. If s and t are arbitrary, then, setting $x = s$, $z = t$, we get solutions

$$\begin{bmatrix} s \\ 3s + 2t - 6 \\ t \end{bmatrix} \qquad s \text{ and } t \text{ arbitrary}$$

Moreover, each solution arises in this way, so we have determined all solutions. Of course we could have solved for x.

$$x = \frac{1}{3}(y - 2z + 6)$$

Then, if we take $y = p$, $z = q$, the solutions are represented as follows:

$$\begin{bmatrix} \frac{1}{3}(p - 2q + 6) \\ p \\ q \end{bmatrix} \qquad p \text{ and } q \text{ arbitrary}$$

The same family of solutions can "look" quite different! □

When only two variables are involved, the solutions to systems of linear equations can be described geometrically because the graph of a linear equation $ax + by = c$ is a straight line. Moreover, a point $P(s, t)$ with coordinates s and t lies on the line if and only if $as + bt = c$—that is, when $\begin{bmatrix} s \\ t \end{bmatrix}$ is a solution to the equation. Hence solutions $\begin{bmatrix} s \\ t \end{bmatrix}$ to a *system* of linear equations correspond to the points $P(s, t)$ that lie on *all* the lines in question. In particular, if the system consists of just one equation (as in Example 3) there must be infinitely many solutions because there are infinitely many points on a line. If the system has two equations, there are three possibilities for the corresponding straight lines.

1. They intersect in a single point. Then the system has a *unique solution* corresponding to that point.
2. They are parallel (and distinct) and so do not intersect. Then the system has *no solution*.
3. They are identical. Then the system has *infinitely many solutions*—one for each point on the (common) line.

These three situations are illustrated in Figure 1.1. In each case the graphs of two specific lines are plotted and the corresponding equations indicated. In the last case, the equations are $3x - y = 4$ (treated in Example 3) and $-6x + 2y = -8$, which have identical graphs.

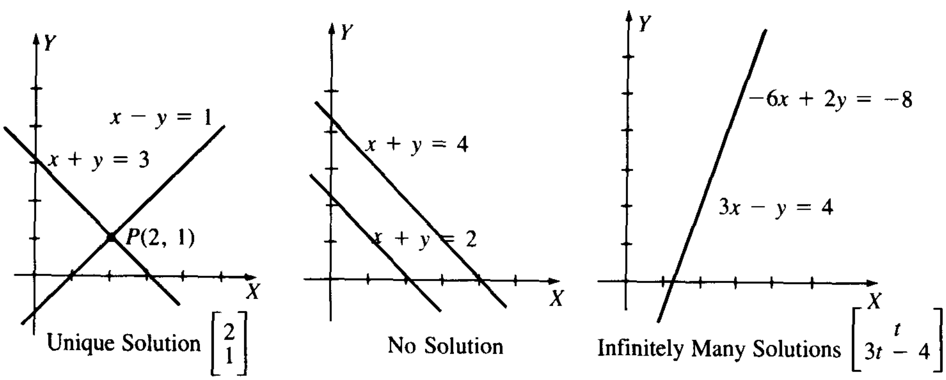

FIGURE 1.1

A similar situation occurs when three variables are present. The graph of an equation $ax + by + cz = d$ can be shown to be a plane (provided not all of a, b, and c are zero) and to consist of all points $P(r, s, t)$ in space such that $\begin{bmatrix} r \\ s \\ t \end{bmatrix}$ is a solution to the equation. Hence, as we found for lines, this plane provides a "picture" of the set of solutions of the equation. In particular, the solutions to a system of three linear equations in three variables correspond to the points common to all three planes. The same possibilities arise as before: namely, no solution, a unique solution, or infinitely many solutions (see Figure 1.2). In fact, it is not difficult to show that one