MATHEMATICS APPLIED to Deterministic Problems in the Natural Sciences

C. C. LIN
L. A. SEGEL

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C. C. LIN *Massachusetts Institute of Technology*

L. A. SEGEL *Rensselaer Polytechnic Institute and Weizmann Institute of Science*

with material on elasticity by

G. H. HANDELMAN *Rensselaer Polytechnic Institute*

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Preface

THIS text, *an introduction to applied mathematics, is concerned with the* construction, analysis, and interpretation of mathematical models that shed light on significant problems in the natural sciences. It is intended to provide material of interest to students in mathematics, science, and engineering at the upper undergraduate and graduate level. Classroom testing of preliminary versions indicates that many such students do in fact find the material interesting and worthy of study.

There is little doubt that a course such as one based on this text should form part of the core curriculum for applied mathematicians. Moreover, in the last few years the professional mathematical community in the United States has emphasized the importance of some exposure to applied mathematics for all mathematics students. This exposure is recommended because of its broadening influence, and (for future university mathematicians) because of the benefits it affords in preparing for the teaching of nonspecialists. As for scientists and engineers, there is often little difference in their theoretical work and that of an applied mathematician, so they should find something of value in the approaches to problems that we espouse.

There are many books that present collections of useful mathematical techniques and illustrate the various techniques by solving classical problems of mathematical physics. Our approach is different. Typically, we select an important scientific problem whose solution will involve some useful mathematics. After briefly discussing the required scientific background, we formulate a relevant mathematical problem with some care. (The formulation step is often difficult. Not many books actually demonstrate this, but we try to give due weight to the challenges involved in constructing our mathematical models.) A new technique may then be introduced to solve the mathematical problem, or a technique known in simpler contexts may be generalized. In most instances we take care to determine what the mathematical results tell us about the scientific processes that motivated the problem in the first place.

We use a "case study" approach by and large. Such an approach is not without disadvantages. No strict logical framework girds the discussion, and the range of applicability of the methods is not precisely delineated. Heuristic and nonrigorous reasoning is often employed, so there is room for doubt concerning the results obtained. But realistic problems often require techniques that cannot at the moment be rigorously justified. There is a stimulating sense of excitement in tackling such problems. Furthermore, mathematicians and scientists frequently use heuristic reasoning and are frequently called upon to determine for themselves whether a method used to solve one problem can be adapted to solve another. Some such experience should be part of each student's education.

This work was given a lengthy title because we wished to make its limitations clear at the outset. A completely balanced introduction to applied mathematics should contain material from the social and managerial sciences, but we have restricted ourselves to the natural sciences. In rough proportion to applied mathematical research, the topics in this volume are drawn mainly from the physical sciences, but there is representation from chemistry and biology. We treat probabilistic models to a lesser extent than would be required in a balanced presentation, and our treatment emphasizes the relationship between probabilistic and deterministic points of view. Our work is also limited in its almost exclusive use of analytical methods; numerical methods are mentioned many times but are treated only briefly.

One reason for restricting the topics covered is the authors' hesitation to tread outside their areas of expertise. Another is the fact that the work is already lengthy, so a wider purview would necessarily be either overlong or superficial. In any case, much further study is required of the aspiring mathematician or scientist—we hope that our work will form a foundation and motivation for some of that study.

STYLE AND CONTENT

In our writing we have striven in most places for careful and detailed exposition, even at the risk of wordiness; for a rigorous proof can be built from its skeleton, but the reasoning of the applied mathematician often can only be mastered if it is fully described.

The nature of applied mathematics precludes an approach that is organized in a tight linear fashion. This has certain disadvantages. But among the compensations is a high degree of flexibility in a book such as this. In particular, there is a large measure of independence among the three parts of the present volume, which are the following:

PART A. An overview of the interaction of mathematics and natural science.

PART B. Some fundamental procedures—illustrated on ordinary differential equations.

PART C. Introduction to theories of continuous fields.

There is considerable further independence within each part—which we have tried to enhance, even at the expense of repetition.

Two volumes have been planned. This first volume provides ample material for a balanced and self-contained introduction to a major part of applied mathematics. The succeeding volume, described briefly in Section 1.1, penetrates further into the subject, particularly in the classical areas of fluid mechanics and elasticity.

The chapter titles, section titles, and subheadings in the table of contents give a good outline of this volume. It is not necessary to read the chapters in the given order. In Part A, for example, the only sequence which must be kept is that of the two chapters on Fourier series. It would be helpful to begin Part B by obtaining some understanding of nondimensionalization and scaling as treated in Chapter 6, but this is not strictly necessary. Chapter 8 can only be appreciated if the preceding two chapters have been covered, but this chapter can be omitted if relatively simple examples of the techniques are deemed sufficient. Chapter 9 is a prerequisite for Chapter 10, but each of the three sections of Chapter 11 is largely independent of the others and of earlier material.

Part C, too, offers various possibilities. For example, one can skip much of the material if one's goal is to obtain just enough understanding of the basic equations to permit formulation of specific problems in one-dimensional elasticity, inviscid flow, and potential theory. Or one can just study the first two sections of Chapter 12, to gain a glimpse of continuum mechanics. (Note: Section 12.1 contains many new ideas in a few pages.)

Some features of our approach are the following.

(a) We proceed from the particular to the general.

(b) For our major examples we attempt to choose physical problems that are important in their own right and also permit the illustration of major mathematical techniques. Thus the Michaelis–Menten kinetics discussed in Chapter 10 are repeatedly referred to in biochemistry, and a full treatment of the relevant mathematical problem provides an excellent illustration of singular perturbation theory. To give another example, we discuss the stability of a stratified fluid in Chapter 15—both as an illustration of inviscid flow and as a motivation for studying stability theory for a system of partial differential equations.

(c) We make a serious effort to examine the processes of deriving the equations that model certain basic scientific phenomena, rather than merely give plausibility arguments for using such equations. As an illustration of this spirit, we mention that the differential equation which governs mass conservation in a continuum is derived in four different ways in Section 14.1, and several alternative approaches are examined in the exercises of that section. One purpose of such an effort is to engender a secure understanding of the equation in question. Another is to help those readers who might some day wish to derive equations that model a phenomenon which had never before been subjected to mathematical analysis.

(d) New ideas are frequently introduced in extremely simple physical contexts. In Part B, for example, dimensional analysis, scaling, and regular perturbation theory are first met in the context of the problem of a point mass shot vertically from the surface of the earth. The qualitative features of the phenomenon are correctly guessed by most people, and the relevant differential equation is solved exactly in elementary courses. Yet, considerable effort is required to obtain a deep understanding of the problem. This effort is worthwhile because it generates a grasp of concepts that are useful in far more difficult situations.

(e) We try to make explicit various concepts and approaches that are often mastered only by inference over a period of years. Examples are provided by our discussions of the basic simplification procedure and of scaling, in Chapter 6.

(f) In historical remarks we have focused attention on humanistic aspects of science, by emphasizing that the great structures of scientific theories are gradually built by the strivings of many people. To illustrate the ongoing nature of science, we have presented certain plausible theories which are either incorrect (Newton's isothermal speed of sound—Section 15.3) or not fully in accord with observation (the galactic model of Section 1.2), or highly regarded but not yet fully accepted (as in the physiological flow problem discussed in Chapter 8).

(g) Some rather lengthy examples are worked out in detail, e.g., the perturbation calculations in Section 7.2, in response to student objections that they are often asked in exercises and in examinations to solve much harder problems than they have ever seen done in the text.

(h) We have provided a number of exercises to reinforce, test, and extend the reader's understanding. Noteworthy are multipart exercises, often based on a relatively recent journal article, which develop a major point in a step-by-step manner. (An example is Exercise 15.2.10.) Even if a student cannot do one part of the exercise, he can take its result for granted and proceed. Such exercises have been successfully used as the central part of final examinations.

PREREQUISITES

We have assumed that the potential reader has had an introductory college course in physics and is familiar with calculus and differential equations. Only a few exercises require knowledge of complex analysis. We make considerable use of such topics as directional derivatives, change of variables in multiple integrals, line and surface integrals, and the divergence theorem. Often mathematics majors will have taken an advanced calculus course that omits some of these topics, but we have found that such students are sufficiently sophisticated mathematically to be able readily to pick up by themselves what is required. [Potential readers who feel inadequacies in vector calculus and physical reasoning would profit from studying *Div, Grad, Curl, and All That* by H. M. Schey (N.Y., Norton, 1973).]

RELATIONSHIPS BETWEEN THIS TEXT AND VARIOUS COURSES

Historically, this book grew out of the union of two courses. The first was *Foundations of Applied Mathematics* introduced by G. H. Handelman at Rensselaer around 1957. (A precursor of this course was taught by A. Schild and Handelman at Carnegie Institute of Technology, now Carnegie-Mellon

University.) A second course, *Introduction to Applied Mathematics*, was introduced by C. C. Lin at Massachusetts Institute of Technology around 1960. These courses have been taught annually, many times, by the present authors in their respective institutions. In recent years preliminary drafts of the present work have been used as text material. Such drafts have also been used in applied mathematics courses taught by P. Davis at Worcester Polytechnic Institute, D. Drew at New York University, and D. Wollkind at Washington State University. Considerable improvements in the draft text have resulted; the authors welcome further suggestions from users of the printed work.

ACKNOWLEDGMENTS

The work was jointly planned. One of us (C.C.L.) wrote the initial draft of Part A, with the exception of Section 1.2, and also Chapter 16. The other of us (L.A.S.) drafted the remainder of the book, with the exception of Chapter 12, written by G. H. Handelman. There was considerable consultation on revisions. L.A.S. was responsible for the final editorial work.

In writing this book, we have drawn on a background for which we are deeply indebted to our families, colleagues, teachers, and students, and to the writers of numerous other books. We are grateful to G. H. Handelman for showing, from classroom experience, that an introduction to continuum mechanics is probably best made by starting with one-dimensional problems—and for writing out his approach as Chapter 12. Among many who have made useful suggestions concerning this volume, we must single out Roy Caplan, Paul Davis, Donald Drew, William Ling, Robert O'Malley, Jr., Edward Rothstein, Terry Scribner, Hendrick Van Ness, and especially Edith Luchins. Numbers of secretaries have performed yeoman service. The publishers have also been most helpful, particularly our editors Everett Smethurst and Elaine Wetterau.

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C. C. L.
L. A. S.

Conventions

EACH chapter is divided into several sections (e.g., Section 5.2 is the second section of Chapter 5). Equations are numbered consecutively within each section. Figures and tables are numbered consecutively within each chapter.

When an equation outside a given section is referred to, the section number precedes the equation number. Thus "Equation (6.3.2)" [or (6.3.2)] refers to the second numbered equation of Section 6.3. But if this equation were referred to within Section 1 of Chapter 6, then the chapter number would be assumed and the reference would be to "Equation (3.2)" [or (3.2)]. The fourth numbered equation in Appendix 3.1 is denoted by (A3.1.4).

A double dagger‡ preceding an Exercise, or a part thereof, signifies that a hint or an answer will be found in the back of this volume.

The symbol □ signifies that the proof of a theorem has concluded.

The succeeding volume is referred to as "II."

A brief bibliography of books useful to beginning applied mathematicians can be found at the end of this volume. When reference is made to one of these books, the style "Smith (1970)" is employed.

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