Path integral methods in quantum field theory

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PATH INTEGRAL METHODS IN QUANTUM FIELD THEORY

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Preface

The use of functionals in the quantisation of relativistic local field theories has a long history, going back to the work of Schwinger and Symanzik in the 1950s. As exemplified by the generating functionals for Green functions, they can embody the canonical Hamiltonian results in a very convenient way (the Dyson–Schwinger equations). By the end of the 1960s the use of functionals had become standard practice (see Fried's book of 1972 with essentially the same title as this) but it was by no means obligatory. Straightforward manipulation of Feynman diagrams was often sufficient.

The renaissance of field theory in the 1970s after the failure of the pure S-matrix approach was based on models that were much more complicated than those used hitherto. Non-Abelian gauge theories, spontaneous symmetry breaking, supersymmetry (to name but three essential steps) required a reappraisal of tactics. At the same time the quantities that needed to be calculated changed in character. Rather than S-matrix elements it was important to determine free energies, tunnelling decay-rates, critical temperatures, Wilson loops, etc.

These quantities, with a natural definition via the standard functionals of the theory, required an approach based on them. The ingredient that gave the functional approach the additional power to cope with the new complexity was the side-stepping of canonical Hamiltonian methods by the use of path integrals to represent the functionals.

Path integrals (the terminology for functional integrals, derived from their quantum mechanical origin) had been introduced by Dirac in the 1930s. However, their mathematical fuzziness (in contrast to the precise Euclidean Wiener integrals) had discouraged their serious application in quantum theory. Nonetheless, despite the absence of a mathematical

definition it became apparent that path integrals in field theory were ideally suited to

(i) implement the symmetries of the theory directly,
(ii) incorporate constraints simply,
(iii) explore field topology,
(iv) isolate relevant dynamical variables,
(v) describe non-zero temperature.

These were key ingredients in model-making for unified theories, and by the late 1970s a working knowledge of functional integrals had become extremely useful for most field theorists.

At the same time a further boost to path integrals arrived with the availability of large main-frame computers to perform numerical simulations of them. By successfully approximating infinite-dimensional functional integrals by mathematically run-of-the-mill finite-dimensional ones they reinforced the feeling that mathematical impreciseness was not a serious problem.

The cumulative effect has been to make some understanding of functional integrals essential for field theorists. In this monograph I aim to show that despite the lack of mathematical foundations the path integral functionals provide, in their simplest expression as 'integrals' over field configurations, a ready tool for analytic approximations. In particular, by working with the naive formalism, we are able to preserve the classical intuition that is often the starting-point for model-making, and which is lost if we attempt to enforce mathematical probity from the beginning. This viewpoint was stressed in the formulation of path integrals and is the most compelling reason for evading mathematical nicety.

Most of the simple applications discussed in the text only involve the ability to perform a generalised Gaussian integral. (Some only require an identification of a Gaussian.) Despite this we can attack problems ranging from the existence of phase transitions in the early universe to the nature of the divergence of the series expansion of quantum electrodynamics in the fine-structure constant.

Nonetheless, some caution is required. For all the plausibility of the formalism, cavalier mathematical manipulations can let us down. However, forewarned is largely forearmed. The largest individual chapter (chapter 6) tries to provide a simple idea of the main hazards when writing down expressions that portray more how we would like the theory to be, rather than how it is. The reader only interested in applications can bypass this self-contained discussion.

An abiding problem in writing a methods book is that of maintaining

sufficient brevity for the diversity not to become ponderous. Even then the list of applications is nowhere near exhaustive (e.g. supersymmetry is hardly considered). In general each chapter describes a particular analytic application of functional integrals with, wherever possible, a non-trivial application. Necessarily these examples, to be tractable, contain over-simplifications. In the sense of real physics (i.e. numbers) I shall often get no further than the foothills. To continue the analogy the aim has been to provide a route map, rather than a detailed description of the topography. Such details are best provided by review articles, to which this book is a preparatory course. This does lead to a problem with references. Since the number of physicists involved in way-marking a particular trail is large – a review article can easily have more than a hundred references – the simplest approach would be to reference the most appropriate review and leave it at that. I have tried to strike a balance between this minimal approach and the impossible task of quoting all significant papers. Inevitably, many authors who have made significant contributions to particular topics have not been quoted explicitly, and will only appear as references of references. To these I apologise.

The genesis of this book was as a lecture course given to first-year postgraduate students of mathematical physics at Imperial College. The course was given in the second half of the year, parallel with an advanced course in quantum field theory. I make two assumptions of the reader. The first is a rudimentary understanding of Feynman diagrams in relativistic quantum field theory, and the need for renormalisation. (However, the most difficult calculations will only involve a single loop.) The second is a familiarity with the basic ideas of realistic field theories, unified at the electroweak level at least. The applications will be sufficiently simple that no knowledge of group theory is required beyond the simplest representation theory.

Beyond that, no prior knowledge of functional integration is necessary. This is developed as required from a very primitive level (and usually stays that way). Occasionally it will be necessary to sidestep solved problems so as to keep the discussion succinct, but all crucial points will be covered.

Finally, my views have been shaped by contact with many physicists. In particular I must thank Eduardo Caianiello of Salerno, John Klauder of Bell Laboratories, and my colleagues at Imperial College for helping me acquire such insights as I possess. My especial thanks go to Chris Isham for his helpful and constructive comments on the manuscript of this book.

R. J. R. 1987

Contents

1

Scalar Green functions and their perturbative solutions

In this book we shall be almost exclusively concerned with the interactions of relativistic particles that are the quanta of *elementary* fields. There is some ambiguity in the definition of 'elementary', but by it we mean local fields whose propagation and interactions can be described by a local Hamiltonian, or Lagrangian, density. Individual terms in these densities describe the basic transformations that the quanta can undergo. For example, if the classical Lagrangian density for a field A has a quartic gA^4 interaction we assume that, quantum mechanically, one A-particle can turn directly into three (virtual) A-particles. The way in which these virtual particles further split or recombine determines the way in which A-particle interactions take place.

The aim of this first chapter is to indicate how canonical quantisation (i.e. the Hamiltonian formulation) can be reformulated as statements about how particles interact. The quantification of the qualitative statement that one A-particle can turn into three, or whatever, will occur through a set of relations termed the Dyson–Schwinger equations. In our approach these equations will play a critical role in formulating an alternative quantisation of field theory through path integrals. The path integral formulation, rather than the canonical approach, will be at the centre of all our calculational methods.

This will come later. First, we must derive the Dyson–Schwinger equations. Their content is essentially *combinatoric*, by which we mean that it concerns the structure of the interactions, the book-keeping of particle creation and annihilation. This can be seen most simply in the theory of a neutral scalar field $A(x)$ like the one mentioned above. Although such a theory has no physical significance its combinatoric structure, as expressed through its Green functions, is exemplary of more

realistic theories. For this reason, in this and the next several chapters we shall examine the properties of scalar fields alone.

By and large the tactics that we develop can, and will, be extended to more realistic theories containing leptons, quarks, gluons, etc. There is just one initial problem to be sidestepped. Surprisingly, for the relevant case of $n = 4$ space-time dimensions the prototypical gA^4 theory is most likely to be trivial (Frölich 1982). That is, after taking self-interactions into account, A-particles do not interact. This screening of the A-field 'charge' g by quantum fluctuations fortunately does not spill over to realistic models. Nonetheless, it is something of an embarrassment to the perturbative tactics that we shall adopt initially. We shall avoid the problem by restricting ourselves to known non-trivial theories (e.g. gA^4 in $n < 4$ dimensions).

1.1 Quantisation of a scalar field

With the above caveat in mind, consider the theory of a single classical c-number scalar field A in n space-time dimensions with Lagrangian density

$$\mathscr{L}(\partial_\mu A, A) = \tfrac{1}{2}\partial_\mu A \partial^\mu A - \tfrac{1}{2}m^2 A^2 - U(A) \tag{1.1}$$

$U(A)$ describes the field self-interaction (usually $gA^4/4!$).

We have adopted a contravariant–covariant Lorentz-vector notation in which $x^\mu = (t, \mathbf{x})$ or, more conveniently, $x^\mu = (x_0, \mathbf{x})$ and $\partial_\mu = \partial/\partial x^\mu$. The contravariant co-ordinate vector $x_\mu = (x_0, -\mathbf{x}) = g_{\mu\nu}x^\nu$, where $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski metric.

The classical Hamiltonian density is constructed from $\mathscr{L}$ as

$$\mathscr{H}(\pi, A) = \pi\partial_0 A - \mathscr{L}(\pi, \nabla A, A) \tag{1.2}$$

with

$$\pi = \partial\mathscr{L}/\partial(\partial_0 A) = \partial_0 A \tag{1.3}$$

the conjugate field variable. The Hamiltonian for the theory is thus

$$H = \int \mathrm{d}\mathbf{x}\,\mathscr{H} = \int \mathrm{d}\mathbf{x}\,[\tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\nabla A)^2 + \tfrac{1}{2}m_0^2 A^2 + U(A)] \tag{1.4}$$

We wish to quantise this classical system in the Heisenberg representation, generalising the results of quantum mechanics in the simplest possible way. The dynamics are determined from the canonical Heisenberg equations of motion

$$\partial_0 \hat{A}(x) = \mathrm{i}\hbar^{-1}[\hat{H}, \hat{A}(x)], \quad \partial_0 \hat{\pi}(x) = \mathrm{i}\hbar^{-1}[\hat{H}, \hat{\pi}(x)] \tag{1.5}$$

The circumflexes denote operators acting on the Hilbert space of the theory. These equations are augmented by the equal-time commutation relations (ETCRs)

$$[\hat{A}(x_0, \mathbf{x}), \hat{\pi}(x_0, \mathbf{y})] = i\hbar\delta(\mathbf{x} - \mathbf{y}),$$
$$[\hat{A}(x_0, \mathbf{x}), \hat{A}(x_0, \mathbf{y})] = 0 = [\hat{\pi}(x_0, \mathbf{x}), \hat{\pi}(x_0, \mathbf{y})] \tag{1.6}$$

that are the causal generalisation of the canonical commutation relations $[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$, etc. of quantum mechanics. If the ETCRs are true at one time they are true at all times because of the Heisenberg equations (1.5) and so play the role of boundary conditions.

With no problems of ordering $\hat{A}$ and $\hat{\pi}$ because of the simple choice of Hamiltonian, equations (1.5) become (on using (1.6))

$$\partial_0 \hat{A} = (\partial \mathscr{H}/\partial \pi)_{\hat{A},\hat{\pi}}, \quad \partial_0 \hat{\pi} = -(\partial \mathscr{H}/\partial A)_{\hat{A},\hat{\pi}} \tag{1.7}$$

That is, the Heisenberg fields formally satisfy Hamilton's equations. (The suffix $\hat{A}$ denotes the replacement of the c-number field A by the operator-valued $\hat{A}$ in the c-number bracket. The suffix $\hat{\pi}$ has a similar meaning. Although a little cumbersome, variants on this notation will prove very economical.) Combining equations (1.7) shows them to be equivalent to the operator-valued Euler–Lagrange equations

$$0 = [\delta S[A]/\delta A(x)]_{\hat{A}} = [\partial \mathscr{L}/\partial A(x) - \partial_\mu(\partial \mathscr{L}/\partial(\partial_\mu A(x)))]_{\hat{A}}$$
$$= -(\Box_x + m^2)\hat{A}(x) - U'(\hat{A}(x)) \tag{1.8}$$

obtained by variation of the classical action functional

$$S[A] = \int \mathrm{d}x\, \mathscr{L}(\partial_\mu A, A) \tag{1.9}$$

($\Box$ is shorthand for $\partial_\mu \partial^\mu$ and $\mathrm{d}x$ shorthand for $\mathrm{d}^n x$).

To clarify the notation of (1.8) some comment on functionals and their differentiation is necessary. A *functional* $F[A]$ of a real classical field $A(x)$ is a rule that associates a number (generally complex) to each real *configuration* $A(x)$. At their simplest, like the action functional, functionals are integrals of functions, for example

$$F[A] = \int \mathrm{d}x\, f(A(x)) \tag{1.10}$$

(although we shall develop more complicated ones). We adopt the convention of using square brackets to enclose the arguments of functionals, curved brackets the arguments of functions. The operation of

functional differentiation, denoted $\delta F[A]/\delta A(y)$, is constructed to do the obvious: for $F[A]$ of (1.10), just depending on A and not its derivatives,

$$\delta F[A]/\delta A(y) = f'(A(y)) \tag{1.11}$$

where $f'(A) = \mathrm{d}f/\mathrm{d}A$. This can be understood as arising from the definition

$$[\delta/\delta A(y), A(x)] = \delta(x - y) \tag{1.12}$$

on taking the differentiation through the integral.

Alternatively, again assuming that $\delta F/\delta A$ exists, we can generalise the lay definition of the derivative of a function to a functional derivative as

$$\delta F[A]/\delta A(y) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [F[A'] - F[A]] \tag{1.13}$$

where $A'(x) = A(x) + \varepsilon\delta(x - y)$. The result (1.11) for F of (1.10) follows equally easily.

Returning to the problem of quantisation we have seen that the quantised theory is formulated by the classical action equation (1.8) for the Heisenberg fields, plus the ETCRs (1.6). (In fact, both these ingredients can be derived from a single action principle (Schwinger, 1951).) However, these operator equations are not convenient for calculational purposes as they stand. In order to turn them into statements about observables for the A-field they need to be sandwiched between suitable states. The equations then become relations between transition elements, the most useful of which are the Green functions of the theory.

1.2 Green functions

The m-leg scalar Green function $G_m(x_1 x_2 \ldots x_m)$ is defined as

$$G_m(x_1 x_2 \ldots x_m) = \langle 0|T(\hat{A}(x_1)\hat{A}(x_2)\ldots\hat{A}(x_m))|0\rangle \tag{1.14}$$

where $|0\rangle$ is the ground state of $\hat{H}$ and T denotes time-ordering of the A-fields. This ordering arranges the fields in decreasing time, reading from left to right (remember that $\hat{A}$ commutes with itself at equal times). We assume for the moment that $|0\rangle$ is the *unique* ground state of $\hat{H}$, a property that is inevitable in quantum mechanics but not in field theory.

We represent $G_m(x_1 x_2 \ldots x_m)$ diagrammatically as

The Green functions themselves are not measurable quantities and our organisation of them will be geared to the physical observables that we wish to evaluate. Fifteen years ago there would have been no doubt that the relevant quantities were cross-sections, particle multiplicities, etc. As a result we would have used so-called reduction formulae (Lehmann, Symanzik & Zimmermann, 1955) to turn the G_m into physical transition elements (S-matrix elements) of the A-particles. For example, G_m can be transmuted into a scattering amplitude for two A-particles colliding to produce $(m-2)$ A-particles in the final state. See the standard texts (e.g., Itzykson & Zuber (1980)), for details.

The developments in quantum field theory over the last decade have given us an additional set of quantities to calculate, and our interests will lie with these. As examples for our simple scalar theory, we might wish to calculate energy densities (for determining the existence of symmetry breaking), critical temperatures (should there be a finite temperature phase transition) and metastable vacuum decay rates (as an aid to early-universe calculations). In fact, in later chapters we shall calculate all three. As will be seen, the most useful combination of Green functions in handling problems of this type is their 'generating functional', which we will now construct.

Given a sequence of numbers $g_0, g_1, g_2, \ldots$ the most economical way to encompass them is through their *generating function*

$$z(j) = \sum_{m=0}^{\infty} g_m j^m / m! \tag{1.15}$$

If we know $z(j)$ the g follow directly as its derivatives at $j = 0$. Similarly, the *generating functional* $Z[j]$ for the scalar Green functions G_m is defined by

$$Z[j] = \sum_{m=0}^{\infty} (i\hbar^{-1})^m (m!)^{-1} \times \int \mathrm{d}x_1 \ldots \mathrm{d}x_m j(x_1) \ldots j(x_m) G_m(x_1 x_2 \ldots x_m) \tag{1.16}$$

where $j(x)$ is an arbitrary c-number function. For each $j(x)$, $Z[j]$ is a complex number.

(We have explicitly introduced a factor of $i\hbar^{-1}$ with each $j(x)$. The benefits of doing this are not yet clear, but it must be apparent that the constraints imposed on $Z[j]$ by the Heisenberg equations and the ETCRs will contain factors of $i\hbar$. The choice of coefficient in (1.16) makes for the most concise formulation in the long run.)

To extract the G_m we functionally differentiate with respect to j at $j(x) = 0$, whence

$$G_m(x_1 \ldots x_m) = (-i\hbar)^m \delta^m Z[j]/\delta j(x_1) \ldots \delta j(x_m)|_{j=0} \tag{1.17}$$

In particular, from the definition (1.14) for the G_m, Z can be summed to the compact form

$$Z[j] = \langle 0|T\left(\exp\left[i\hbar^{-1}\int \mathrm{d}x\ j(x)\hat{A}(x)\right]\right)|0\rangle \tag{1.18}$$

That is, $j(x)$ can be interpreted as a source coupled to the A-field.

If we know $Z[j]$ we know everything about the quantum field theory. We have already anticipated its role as a calculational tool. In addition we shall find that it provides for an elegant method of quantisation.

The series (1.16) is represented diagrammatically in fig. 1.1, where the cross-ended line ——×$_x$ corresponds to the multiplication by $i\hbar^{-1}j(y)$ of ——$_y$, followed by integration over y. That is, Z describes the way a source can create particles from the vacuum and return them to it.

The normalisation $\langle 0|0\rangle = 1$ implies $Z[0] = 1$. The symmetry of the source term $j(x_1)j(x_2)\ldots j(x_m)$ in (1.17) under coordinate interchange enforces the Bose symmetry

$$G_m(x_1 x_2 \ldots x_m) = G_m(x_2 x_1 \ldots x_m) \tag{1.19}$$

of the Green functions. Furthermore, from the translational invariance of $|0\rangle$ it follows that, for arbitrary a^μ

$$G_m(x_1 x_2 \ldots x_m) = G_m(x_1 + a, x_2 + a, \ldots x_m + a) \tag{1.20}$$

i.e. G_m only depends on the differences $x_i - x_j$ of co-ordinates.

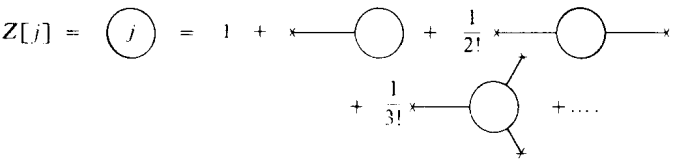

Fig. 1.1

1.3 The Symanzik construction for $Z[j]$

As matrix elements of fields, the Green functions will be constrained by the classical action principle plus the ETCRs. In turn $Z[j]$ will also be restricted. In order to construct $Z[j]$ we follow Symanzik's approach

(Symanzik, 1954) for determining the equation enforced upon $Z[j]$ by canonical quantisation.

To simplify the notation we define $\hat{E}(x_0', x_0)$ by

$$\hat{E}(x_0', x_0) = T\left[\exp\left(i\hbar^{-1}\int_{x_0}^{x_0'} dy_0 \int \mathbf{dy}\, j(y_0, \mathbf{y})\hat{A}(y_0, \mathbf{y})\right)\right] \quad (1.21)$$

$Z[j]$ can be written as

$$Z[j] = \langle 0|\hat{E}(\infty, -\infty)|0\rangle = \langle 0|\hat{E}(\infty, x_0)\hat{E}(x_0, -\infty)|0\rangle \quad (1.22)$$

for any given x_0. It follows that, if $x^\mu = (x_0, \mathbf{x})$,

$$(-i\hbar\delta/\delta j(x))^p Z[j] = \langle 0|T(\hat{A}(x)^p \exp i\hbar^{-1}\int \mathbf{dy}\, j(y)\hat{A}(y))|0\rangle$$

$$= \langle 0|\hat{E}(\infty, x_0)\hat{A}(x)^p \hat{E}(x_0, -\infty)|0\rangle \quad (1.23)$$

insofar as the right-hand side of (1.23) is defined. (This qualification reflects the fact that the relations (1.6) show $\hat{A}$ to be an operator-valued distribution. As a result $\hat{A}^p$ is not defined, in general. However, in practice a definition can be postponed until the last moment.)

Equation (1.23) enables us to rewrite the identity

$$0 = \langle 0|\hat{E}(\infty, x_0)(-\delta S/\delta A(x))_{\hat{A}} E(x_0, -\infty)|0\rangle$$

$$= \langle 0|\hat{E}(\infty, x_0)[(\Box_x + m^2)\hat{A}(x) + U'(\hat{A}(x))]\hat{E}(x_0, -\infty)|0\rangle \quad (1.24)$$

as

$$0 = [(\Box_x + m^2)(-i\hbar\delta/\delta j(x)) + U'(-i\hbar\delta/\delta j(x))]Z[j]$$
$$+ \langle 0|\hat{E}(\infty, x_0)\hat{c}_0^2\hat{A}(x)E(x_0, -\infty)|0\rangle$$
$$- \hat{c}_0^2\langle 0|\hat{E}(\infty, x_0)\hat{A}(x)\hat{E}(x_0, -\infty)|0\rangle \quad (1.25)$$

The difference of the last two terms is calculated trivially. Since $\hat{A}$ commutes with itself at equal time

$$\hat{c}_0\langle 0|\hat{E}(\infty, x_0)\hat{A}(x_0, \mathbf{x})E(x_0, -\infty)|0\rangle$$
$$= \langle 0|\hat{E}(\infty, x_0)\hat{\pi}(x_0, \mathbf{x})\hat{E}(x_0, -\infty)|0\rangle \quad (1.26)$$

However, differentiating again gives

$$\partial_0^2\langle 0|\hat{E}(\infty, x_0)\hat{A}(x_0, \mathbf{x})\hat{E}(x_0, -\infty)|0\rangle$$
$$= \langle 0|\hat{E}(\infty, x_0)\hat{c}_0\hat{\pi}(x_0, \mathbf{x})E(x_0, -\infty)|0\rangle$$
$$- i\hbar^{-1}\langle 0|\hat{E}(\infty, x_0)\left[\int \mathbf{dy}\, j(x_0, \mathbf{y})\hat{A}(x_0, \mathbf{y}), \hat{\pi}(x_0, \mathbf{x})\right]E(x_0, -\infty)|0\rangle$$
$$= \langle 0|\hat{E}(\infty, x_0)\hat{c}_0^2\hat{A}(x)\hat{E}(x_0, -\infty)|0\rangle + j(x)\langle 0|E(\infty, -\infty)|0\rangle \quad (1.27)$$