Non-linear differential equations of higher order

R. Reissig G. Sansone R. Conti

Non-linear differential equations of higher order

R. Reissig

Professor of Mathematics
Ruhr-Universität Bochum
Deutschland

G. Sansone

Professor of Mathematics
Università degli Studi Firenze
Italia

R. Conti

Professor of Mathematics
Università degli Studi Firenze
Italia

NOORDHOFF INTERNATIONAL PUBLISHING
LEYDEN

Preface

In the preface to our work "Qualitative Theorie nichtlinearer Differentialgleichungen" (Edizioni Cremonese, Rome 1963) we have sketched the genesis of this theory and have mentioned some basic text-books and monographs. It is a sign of the steadily growing importance and rapidly progressing development of the theory of non-linear differential equations that in the few years that have lapsed again quite a number of excellent books on important branches have been published. As examples we mention:

A. A. ANDRONOV, E. A. LEONTOVICH, I. I. GORDON and A. G. MAYER, *Qualitative theory of dynamical systems of the second order*, Izdat. Nauka, Moscow (1966).

A. A. ANDRONOV, A. A. VITT and S. E. KHAIKIN, *Theory of oscillators*, Fizmatgiz., Moscow (1959). Translation: Pergamon Press, Oxford-New York (1966).

A. BLAQUIÈRE, *Non-linear system analysis*, New York (1966).

W. HAHN, *Stability of motion*, Springer Verlag, Berlin-Göttingen-Heidelberg-New York (1967).

A. HALANAY, *Qualitative theory of differential equations: Lyapunov stability, oscillations, systems with deviating argument* (Roumanian), Ed. Acad. RPR, Bucharest (1963). Translation: Academic Press, New York-London (1966).

J. K. HALE, *Oscillations in nonlinear systems*, McGraw-Hill, New York-London (1963).

PH. HARTMAN, *Ordinary differential equations*, John Wiley & Sons, New York-London (1964).

M. A. Krasnoselskii,
The operator of translation along trajectories of differential equations, Izdat. Nauka, Moscow (1966). Translation: Amer. Math. Soc., Providence, R. I. (1968).
S. Lefschetz,
Stability of nonlinear control systems, Academic Press, New York-London (1965).
I. G. Malkin,
Theory of stability of motion, second ed., Izdat. Nauka, Moscow (1966).
J. L. Massera and J. J. Schaeffer,
Linear differential equations and function spaces, Academic Press, New York-London (1966).
Yu. A. Mitropolskii,
Problems of the asymptotic theory of non-stationary oscillations, Izdat. Nauka, Moscow (1964). Translation: Problèmes de la théorie asymptotique des oscillations non stationnaires, Gauthier-Villars, Paris (1966).
V. A. Pliss,
Non-local problems of the theory of oscillations, Izdat. Nauka, Moscow (1964). Translation: Academic Press, New York-London (1966).
T. Yoshizawa,
Stability theory by Lyapunov's second method, Math. Soc. Japan, Tokyo (1966).

In addition, numerous papers on the behaviour of the solutions of more or less special systems of non-linear differential equations or the properties of general dynamical systems have appeared in the various periodicals or Academy publications. The lion's share of these research results belongs to the USA and USSR, where several new journals have been founded to foster the theory and its applications, journals which now play an important role in the international specialized literature.

A significant part of the publications is concerned with systems of non-linear differential equations of the third and fourth order or certain systems of arbitrary order that are of importance for diverse technical problems. This was already pointed out by G. Sansone in his survey lecture "Non-linear differential equations of the third and fourth order" at the "Equadiff"-Congress (Prague 1962). It was on this occasion that we decided to continue our collaboration and to draw up jointly a report on non-linear differential equations of higher order, and so to close what we felt to be a gap in the literature.

Subsequently we worked out a detailed plan in which we envisaged eight chapters of partly general, partly special nature. True, at the time we did not anticipate that the research on this range of problems would take such a stormy course, which forced us all the time, right to the end of our work, to amplify and bring to the latest state completed parts of the manuscript. Even so we must emphasize that we have not striven for a complete survey of the literature.

In Chapter 1 we treat some ideas and methods of the qualitative theory that are suitable for the study of stability and boundedness of the solutions of ordinary differential equations; here Lyapunov's direct method stands in the forefront. We have refrained on purpose from providing well-known results that are easily accessible in text books or to repeat the presentation in our earlier monograph (which is not assumed to be known, but may be rather useful). Instead we discuss some contributions in the modern specialized literature, which we regard as noteworthy for theoretical or practical reasons and which provide a good insight into the trend of arguments of the direct method. In addition, those theorems that can be counted among the classical content of the theory and are applied time and again in specific investigations are quoted explicitly and explained in detail where necessary.

In this chapter we also deal with the existence of periodic solutions of autonomous and non-autonomous systems. We concern ourselves extensively with the second case, because here recent years have seen the growth of interesting methods of functional analysis for existence proofs, which are capable of further development.

Chapter 2 is also devoted to the general theory; we begin with comparison theorems whose purpose it is to reduce the study of the global behaviour of the solutions to the treatment of a simpler type of differential equations. Then we turn to the problem of judging stability on the basis of the system of first (or *m*-th) approximation, and we give the classical results of LYAPUNOV and MALKIN in the most important critical cases.

A multitude of contributions to this theme published in recent years can be found in the list of references at the end of the chapter. A discussion of these works was not possible within the framework we have set for the book.

Chapter 3 contains qualitative studies of the trajectories of autonomous systems in the neighbourhood of an isolated singular point; some more detailed investigations concern systems of the third order.

The next three chapters are of a more specialized character: here we prove theorems on stability and boundedness as well as theorems on the existence of periodic solutions for certain non-linear differential equations of the third and fourth order which one meets frequently in the recent literature; we have given an account of the works of numerous authors, and we are of the opinion that these chapters can claim completeness.

In Chapter 7 we consider autonomous systems with non-linearities in which the variables occur separated, and we state Aizerman's problem; then we follow the arguments set forth by KRASOVSKII, MALKIN, TUZOV, PLISS and others, to solve the problem for certain systems of the second and third order. We believe that the chapter can be very instructive and useful owing to the uniform presentation of these studies.

Chapter 8 turns to systems of Lur'e's type, which play a role in control theory. The importance of such systems was pointed out by MINORSKY on several occasions, and was also emphasized by LEFSCHETZ in his book quoted above. We throw light on the connection between the systems of direct and of indirect control, we then derive some sufficient criteria for stability, and above all we deal with the celebrated theorems of POPOV; we prove them in detail, following ideas of POPOV, AIZERMAN and GANTMAKHER, and CORDUNEANU, and we show to what extent the conditions contained in them are decisive for the existence of certain Lyapunov functions solving the problem of stability.

We have drawn this chapter on a wide canvas, so that readers interested in automatic control but without predilection for mathematical proofs can also follow easily the arguments and calculations.

At the end of each chapter there is a comprehensive list of references. Almost all chapters also contain, apart from numerous results of other authors on which we report in a free presentation and which occasionally we modify to suit our requirements, our own investigations on the subject in question. Besides, we have been able to simplify many a proof or to shorten it on the basis of preceding discussions.

We hope that we have succeeded in giving a well-rounded report with a distinctive flavour, of interest not only to the mathematician occupied with non-linear differential equations, but appealing also to wider circles, for example, applied mathematicians, physicists and engineers who deal with questions of control or with problems of non-linear mechanics and have to rely on the mathematical theory.

We should be pleased if incidentally we have helped in giving their due attention to some papers that for linguistic or other reasons have so far not become known sufficiently well.

Finally, it is our wish that this volume achieves its proper aim of coordinating and describing numerous results spread over the literature under a uniform point of view, of helping the reader in his orientation, and on top of this, of providing stimuli and thereby contributing to the pursuit of pure and applied mathematics.

R. Reissig – G. Sansone – R. Conti

TABLE OF CONTENTS

Chapter 3

The behaviour of the trajectories of autonomous systems

Chapter 4

The equation of type $x''' + ax'' + \varphi(x, x') + f(x) = p(t)$

Chapter 5

The equation of type $x''' + \psi(x, x')x'' + \varphi(x') + f(x) = p(t)$ and other equations

Chapter 6

The equation of type $x'''' + \chi(x'')x''' + \psi(x')x'' + \varphi(x') + f(x) = p(t)$ and more general types

Chapter 7

Systems of arbitrary order Non-linearities with separate variables

Chapter 8

n-th order systems of Lur'e type

CHAPTER 1

General methods of the qualitative theory

In recent years numerous methods have been developed to investigate the qualitative behaviour of solutions of ordinary differential equations; here we discuss in the first instance some particularly interesting methods which are not generally known, and (for the sake of completeness) those main theorems of the qualitative theory that are applied in subsequent chapters in the treatment of special systems (for a more detailed study see, for example, Reissig-Sansone-Conti [1]). We are interested in the stability and boundedness problem as well as the question of the existence of periodic solutions. A number of comparison theorems will be set forth in Chapter 2.

1.1. Some stability theorems

The study of automatic control processes was the starting point for the notion of asymptotic stability "in the large" (or globally) of the zero solution of a differential equation

$$x' = f(t, x) \tag{1.1}$$

$$[f(t, 0) = 0].$$

Let t denote a real variable in the interval $E^1{}_+ : 0 \leq t < +\infty$, where x is a point in E^n, the Euclidean space of real n-vectors

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

with the norm $|x| = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$.

Here $f(t, x)$ is a function in E^n with arguments in $E^1{}_+ \times E^n$; it is assumed to be sufficiently regular so that for every point $(t_0, x_0) \in E^1{}_+ \times E^n$ the existence and local uniqueness (to the right) of the solution $x(t)$ of (1.1) with the initial condition $x(t_0) = x_0$ is guaranteed. If necessary, we denote this solution by $x(t; t_0, x_0)$, even if x for every initial point (t_0, x_0) is only defined for $t \geqq t_0$ as a single-valued function of t.

We say that the zero solution of (1.1) is *attractive in the large* if

$$\lim_{t \to +\infty} x(t; t_0, x_0) = 0, \quad (t_0, x_0) \in E^1{}_+ \times E^n. \tag{1.2}$$

We call it *asymptotically stable in the large* (or *globally asymptotically stable*) if it is attractive in the large and, in addition, stable in the sense of Lyapunov (weakly stable), that is,

$$|x(t; t_0, x_0)| \leqq \varepsilon \text{ for } t \geqq t_0, \text{ in case } |x_0| \leqq \delta(\varepsilon, t_0).$$

This definition was introduced by Barbashin-Krasovskii [1]; the same authors ([2], see also Massera [1]) have formed the following narrower notion:

The zero solution of (1.1) is called *uniformly asymptotically stable in the large* if there exist a function $\sigma(r)$ defined for $r \geqq 0$, continuous and increasing, with $\sigma(0) = 0$, and a function $T(r, \varepsilon)$ defined for $r \geqq 0$, $\varepsilon > 0$, continuous and positive, so that

$$|x(t; t_0, x_0)| \leqq \sigma(r_0) \text{ for } t_0 \leqq t, \ |x_0| \leqq r_0$$

(that is, uniform stability)
and

$$|x(t; t_0, x_0)| \leqq \varepsilon \text{ for } t_0 + T(r_0, \varepsilon) \leqq t, \ |x_0| \leqq r_0.$$

Massera ([1], Theorem 21) has proved:

THEOREM 1.1. *If $f(t, x) = g(x)$ is independent of t, then global asymptotic stability of the zero solution of the autonomous equation*

$$x' = g(x) \quad [g(0) = 0] \tag{1.3}$$

implies uniform global asymptotic stability of this solution.

For uniform asymptotic stability in the large there is the following criterion (Barbashin-Krasovskii [2]; Massera [1], Theorem 22; an extension of the criterion to differential equations in a Banach space was given recently by Conti [4]).

THEOREM 1.2. *The zero solution of* (1.1) *is uniformly asymptotically stable in the large if there exists a real function* $V(t, x)$ *defined in* $E^1_+ \times E^n$ *with* $V(t, 0) = 0$ *and with the properties* a) *to* e):

a) $V(t, x)$ *satisfies a local Lipschitz condition;*

b) $V(t, x)$ *is positive-definite, that is,*

$$V(t, x) \geqq a(|x|)$$

for a function $a(r)$ *defined in the interval* $r \geqq 0$, *continuous and strictly increasing, with the initial value* $a(0) = 0$;

c) $V(t, x)$ *is infinitely large, that is,*

$$\lim_{r \to +\infty} a(r) = +\infty;$$

d) $V(t, x)$ *is uniformly small (or "admits an infinitely small upper bound"), that is,*

$$V(t, x) \leqq b(|x|),$$

where the function $b(r)$ *has the same properties as* $a(r)$;

e) *the function*

$$\limsup_{h \to +0} \{V(t + h, x + hf(t, x)) - V(t, x)\}/h$$

is negative-definite, that is,

$$\limsup_{h \to +0} \{V(t + h, x + hf(t, x)) - V(t, x)\}/h \leqq -c(|x|), \tag{1.4}$$

where $c(r)$ *is continuous and strictly increasing, with* $c(0) = 0$.

NOTE If the function $V(t, x)$ satisfies the conditions a), b), d), e), but only for $t \geqq 0$, $|x| \leqq H$, then (uniform) asymptotic stability holds locally, namely for

$$|x_0| \leqq h_0 = b^{-1}(a(H)).$$

If then (1.4) holds only with $c(r) \equiv 0$, we obtain uniform stability, that is,

$$|x(t; t_0, x_0)| \leqq \varepsilon \quad \text{for} \quad t \geqq t_0, \quad \text{in case} \quad |x_0| \leqq \delta(\varepsilon);$$

if the condition d) also fails to hold, then we have weak stability.

If $V(t, x)$ is continuously differentiable in all arguments, then

$$\limsup_{h \to +0} \{V(t+h, x+hf(t, x)) - V(t, x)\}/h = V_t + (V_x, f) = V'(t, x).$$

If $V(t, x)$ satisfies the conditions a), b) as well as d), e), then there exists a continuous, strictly increasing function $\omega(r)$ with $\omega(0) = 0$ such that

$$\limsup_{h \to +0} \{V(t+h, x+hf(t, x)) - V(t, x)\}/h \leqq -\omega(V(t, x)). \tag{1.5}$$

For we have

$$|x| \geqq b^{-1}(V(t, x)), \quad c(|x|) \geqq c(b^{-1}(V(t, x)))$$

and we obtain the relation above with the function $\omega = c(b^{-1})$, which obviously has the requisite properties (see Brauer [1]); conversely, (1.4) follows from (1.5) if we set $c = \omega(a)$.

If there exists a continuously differentiable function $V(t, x)$ satisfying the conditions b) to d) and (1.5) (in place of (1.4)), then it is easy to show that the zero solution of (1.1) is uniformly asymptotically stable in the large; for we compute along an arbitrary non-trivial solution $x(t; t_0, x_0)$:

$$\frac{d}{dt}\int_{V_0}^{V(t,x)} \frac{d\varphi}{\omega(\varphi)} \leqq -1, \quad \int_{a(|x|)}^{b(|x_0|)} \frac{d\varphi}{\omega(\varphi)} \geqq \int_{V(t,x)}^{V_0} \frac{d\varphi}{\omega(\varphi)} \geqq t - t_0,$$

that is,

$$|x(t; t_0, x_0)| \leqq \varepsilon \quad \text{for} \quad t - t_0 \geqq T(|x_0|, \varepsilon);$$

here we can set

$$T = \int_{a(\varepsilon)}^{b(|x_0|)} \frac{d\varphi}{\omega(\varphi)}.$$

In accordance with the explanation given in Theorem 1.2 a continuous function $V(x)$ defined in the domain $0 \leqq |x| \leqq H$ (with $V(0) = 0$) is to be regarded as positive-definite if it admits a continuous strictly increasing function $a(|x|)$ with $a(0) = 0$ as minorant; however, it is customary to regard the function $V(x)$ already as positive-definite if $V(x) > 0$ for $x \neq 0$.

In order to show that the two explanations are equivalent, we have to derive the first from the second.

For this purpose we set

$$\varphi(r) = \inf_{r \leqq |x| \leqq H} V(x)$$

$$[\varphi(0) = 0, \quad \varphi(r) > 0 \quad \text{for} \quad r > 0]$$

and can first give a lower estimate of the function V in terms of the non-decreasing continuous function φ:

$$V(x) \geqq \varphi(|x|).$$

Next we determine for $0 \leqq r \leqq H$

$$a(r) = \frac{r\varphi(r)}{H} \leqq \varphi(r)$$

and then we have for $0 \leqq r' < r'' \leqq H$

$$a(r') = \frac{r'\varphi(r')}{H} \leqq \frac{r'\varphi(r'')}{H} < \frac{r''\varphi(r'')}{H} = a(r'').$$

By Theorem 1.2, a function $V(x)$ defined for all x and continuous, with $V(0) = 0$, is positive-definite and infinitely large if it admits an estimate

$$V(x) \geqq a(|x|),$$

where the function $a(r)$, with $a(0) = 0$, is continuous for $r \geqq 0$ and strictly increasing, with

$$\lim_{r \to \infty} a(r) = \infty.$$

Equivalent to this definition is the following:

$$V(x) > 0 \quad \text{for} \quad x \neq 0, \quad \lim_{|x| \to \infty} V(x) = \infty.$$

To prove this we have again to reduce the second definition to the first.

For this purpose we set, as before,

$$\varphi(r) = \inf_{|x| \geqq r} V(x)$$

and obtain a function, continuous for $r \geqq 0$ and non-decreasing, with

$$\varphi(0) = 0, \quad \varphi(r) > 0 \quad \text{for} \quad r > 0, \quad \lim_{r \to \infty} \varphi(r) = \infty$$

and

$$V(x) \geqq \varphi(|x|) \text{ for all } x.$$

We determine a strictly increasing divergent sequence of numbers $\{\rho_k\}$ satisfying the condition:

$$\varphi(\rho_k) = k, \ \varphi(r) > \varphi(\rho_k) \text{ for } r > \rho_k \ (k = 0, 1, 2, \ldots);$$

here $\rho_0 = 0$.

Finally, we define (for $k \geqq 0$)

$$a(r) = \varphi(\rho_k) + \frac{r - \rho_k}{\rho_{k+1} - \rho_k} [\varphi(r) - \varphi(\rho_k)], \ \rho_k \leqq r \leqq \rho_{k+1}.$$

Then

$$a(r) \leqq \varphi(r)$$

$$[a(\rho_k) = \varphi(\rho_k), \ a(\rho_{k+1}) = \varphi(\rho_{k+1})]$$

and for $\rho_k \leqq r' < r'' \leqq \rho_{k+1}$

$$a(r') \leqq \varphi(\rho_k) + \frac{r' - \rho_k}{\rho_{k+1} - \rho_k} [\varphi(r'') - \varphi(\rho_k)] < a(r'');$$

in addition we obtain

$$\lim_{r \to \infty} a(r) = \lim_{k \to \infty} \varphi(\rho_k) = \infty.$$

Every function $V(x)$, continuous for all x and positive for $x \neq 0$, is uniformly small in the sense of Theorem 1.2; for if we set

$$\psi(r) = \sup_{0 \leqq |x| \leqq r} V(x)$$

and

$$b(r) = \psi(r) e^r,$$

then for all x

$$V(x) \leqq b(|x|) \text{ (strictly increasing).}$$

Consequently we can combine the conditions b) to d) in Theorem 1.2 into the following:

$$V_1(x) \leqq V(t, x) \leqq V_2(x)$$