

# VECTOR & TENSOR ANALYSIS

G. E. HAY

# VECTOR AND TENSOR ANALYSIS

*by*

**G. E. HAY**

*Associate Professor of Mathematics  
University of Michigan*

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# TABLE OF CONTENTS

## CHAPTER I. ELEMENTARY OPERATIONS

	Page
1. Definitions . . . . .	1
2. Addition of vectors . . . . .	2
3. Multiplication of a vector by a scalar . . . . .	4
4. Subtraction of vectors . . . . .	6
5. Linear functions . . . . .	6
6. Rectangular cartesian coordinates . . . . .	7
7. The scalar product . . . . .	10
8. The vector product . . . . .	11
9. Multiple products of vectors . . . . .	15
10. Moment of a vector about a point . . . . .	18
11. Moment of a vector about a directed line . . . . .	20
12. Differentiation with respect to a scalar variable . . . . .	22
13. Integration with respect to a scalar variable . . . . .	25
14. Linear vector differential equations . . . . .	26
Problems . . . . .	28

## CHAPTER II. APPLICATIONS TO GEOMETRY

15. Introduction . . . . .	34
16. Some theorms of plane geometry . . . . .	34
<i>Solid Analytic Geometry</i>	
17. Notation . . . . .	37
18. Division of a line segment in a given ratio . . . . .	38
19. The distance between two points . . . . .	39
20. The area of a triangle . . . . .	40
21. The equation of a plane . . . . .	41
22. The vector-perpendicular from a point to a plane . . . . .	43
23. The equation of a line . . . . .	46

	Page
24. The equation of a sphere . . . . .	49
25. The tangent plane to a sphere . . . . .	50

### *Differential Geometry*

26. Introduction . . . . .	51
27. The principal triad . . . . .	52
28. The Serret-Frenet formulas . . . . .	53
29. Curvature and torsion . . . . .	55
Problems . . . . .	58

## CHAPTER III. APPLICATION OF VECTORS TO MECHANICS

### *Motion of a Particle*

30. Kinematics of a particle . . . . .	62
31. Newton's laws . . . . .	66
32. Motion of a particle acted upon by a force which is a given function of the time . . . . .	68
33. Simple harmonic motion . . . . .	69
34. Central orbits . . . . .	70

### *Motion of a System of Particles*

35. The center of mass of a system of particles . . . . .	72
36. The moments and products of inertia of a system of particles . . . . .	73
37. Kinematics of a rigid body . . . . .	77
38. The time derivative of a vector . . . . .	79
39. Linear and angular momentum . . . . .	80
40. The motion of a system of particles . . . . .	83
41. The motion of a rigid body with a fixed point . . . . .	84
42. The general motion of a rigid body . . . . .	94
Problems . . . . .	97

## CHAPTER IV. PARTIAL DIFFERENTIATION

43. Scalar and vector fields . . . . .	102
44. Directional derivatives. The operator $\text{del}$ . . . . .	102
45. Properties of the operator $\text{del}$ . . . . .	105
46. Some additional operators . . . . .	107

	Page
47. Invariance of the operator $\text{del}$ . . . . .	111
48. Differentiation formulas . . . . .	117
49. Curvilinear coordinates . . . . .	120
50. The expressions $\nabla f$ , $\nabla \cdot \mathbf{b}$ and $\nabla \times \mathbf{b}$ in curvilinear coordi- nates . . . . .	124
Problems . . . . .	127

## CHAPTER V. INTEGRATION

51. Line integrals . . . . .	130
52. Surface integrals . . . . .	134
53. Triple integrals . . . . .	138
Problems . . . . .	139
54. Green's theorem in the plane . . . . .	140
55. Green's theorem in space . . . . .	143
56. The symmetric form of Green's theorem . . . . .	145
57. Stokes's theorem . . . . .	146
58. Integration formulas . . . . .	149
59. Irrotational vectors . . . . .	151
60. Solenoidal vectors . . . . .	152
Problems . . . . .	154

## CHAPTER VI. TENSOR ANALYSIS

61. Introduction . . . . .	157
62. Transformation of coordinates . . . . .	157
63. Contravariant vectors and tensors . . . . .	159
64. Covariant vectors and tensors . . . . .	160
65. Mixed tensors. Invariants . . . . .	161
66. Addition and multiplication of tensors . . . . .	162
67. Some properties of tensors . . . . .	163
68. Tests for tensor character . . . . .	164
69. The metric tensor . . . . .	166
70. The conjugate tensor . . . . .	167
71. Lowering and raising of suffixes . . . . .	169

	Page
72. Magnitude of a vector. Angle between two vectors . . . . .	170
73. Geodesics . . . . .	170
74. Transformation of the Christoffel symbols . . . . .	173
75. Absolute differentiation . . . . .	174
76. Covariant derivatives . . . . .	177
77. The curvature tensor . . . . .	177
78. Cartesian tensors . . . . .	178
79. Oriented cartesian tensors . . . . .	180
80. Relative tensors . . . . .	181
81. Physical components of tensors . . . . .	184
82. Applications . . . . .	186
Problems . . . . .	189

## CHAPTER I

### ELEMENTARY OPERATIONS

1. *Definitions.* Quantities which have magnitude only are called *scalars*. The following are examples: mass, distance, area, volume. A scalar can be represented by a number with an associated sign, which indicates its magnitude to some convenient scale.

There are quantities which have not only magnitude but also direction. The following are examples: force, displacement of a point, velocity of a point, acceleration of a point. Such quantities are called *vectors* if they obey a certain law of addition set forth in § 2 below. A vector can be represented by an arrow. The direction of the arrow indicates the direction of the vector, and the length of the arrow indicates the magnitude of the vector to some convenient scale.

Let us consider a vector represented by an arrow running from a point  $P$  to a point  $Q$ , as shown in Figure 1. The straight line through  $P$  and  $Q$  is called the *line of action* of the vector, the point  $P$  is called the *origin* of the vector, and the point  $Q$  is called the *terminus* of the vector.

To denote a vector we write the letter indicating its origin followed by the letter indicating its terminus, and place a bar over the two letters. The vector represented in Figure 1 is then represented by the symbols  $\overline{PQ}$ . In this book the superimposed bar will not be used in any capacity other than the above, and hence its presence can always

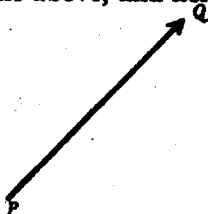


Figure 1



be interpreted as denoting vector character. This notation for vectors is somewhat cumbersome. Hence when convenient we shall use a simpler notation which consists in denoting a vector by a single symbol in bold-faced type. Thus, the vector in Figure 1 might be denoted by the symbol  $\mathbf{a}$ . In this book no mathematical symbols will be printed in bold-faced type except those denoting vectors.\*

The magnitude of a vector is a scalar which is never negative. The magnitude of a vector  $\overline{PQ}$  will be denoted by either  $PQ$  or  $|\overline{PQ}|$ . Similarly, the magnitude of a vector  $\mathbf{a}$  will be denoted by either  $a$  or  $|\mathbf{a}|$ .

Two vectors are said to be equal if they have the same magnitudes and the same directions. To denote the equality of two vectors the usual sign is employed. Hence, if  $\mathbf{a}$  and  $\mathbf{b}$  are equal vectors, we write

$$\mathbf{a} = \mathbf{b}.$$

A vector  $\mathbf{a}$  is said to be equal to zero if its magnitude  $a$  is equal to zero. Thus  $\mathbf{a} = 0$  if  $a = 0$ . Such a vector is called a zero vector.

2. *Addition of vectors.* In § 1 it was stated that vectors are quantities with magnitude and direction, and which obey a certain law of addition. This law, which is called the *law of vector addition*, is as follows.

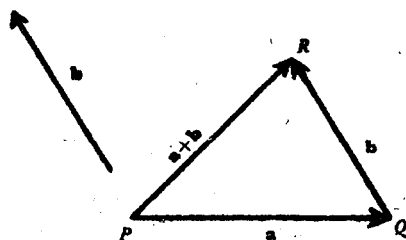


Figure 2

Let  $\mathbf{a}$  and  $\mathbf{b}$  be two vectors, as shown in Figure 2. The origin and terminus of  $\mathbf{a}$  are  $P$  and  $Q$ . A vector equal to  $\mathbf{b}$  is constructed with

\* It is difficult to write bold-faced symbols on the blackboard or in the exercise book. When it is desired to write a single symbol denoting a vector, the reader will find it convenient to write the symbol in the ordinary manner, and to place a bar over it to indicate vector character.

its origin at  $Q$ . Its terminus falls at a point  $R$ . The sum  $\mathbf{a} + \mathbf{b}$  is the vector  $\overrightarrow{PR}$ , and we write

$$\mathbf{a} + \mathbf{b} = \overrightarrow{PR}.$$

**Theorem 1.** Vectors satisfy the commutative law of addition; that is,

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}.$$

**Proof.** Let  $\mathbf{a}$  and  $\mathbf{b}$  be the two vectors shown in Figure 2. Then

$$(2.1) \quad \mathbf{a} + \mathbf{b} = \overrightarrow{PR}.$$

We now construct a vector equal to  $\mathbf{b}$ , with its origin at  $P$ . Its terminus falls at a point  $S$ . A vector equal to  $\mathbf{a}$  is then constructed with its origin at  $S$ . The terminus of this vector will fall at  $R$ , and Figure 3 results. Hence

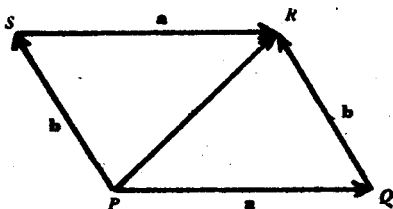


Figure 3

$$(2.2) \quad \mathbf{b} + \mathbf{a} = \overrightarrow{PR}.$$

From (2.1) and (2.2) it follows that  $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ .

**Theorem 2.** Vectors satisfy the associative law of addition; that is,

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}).$$

**Proof.** Let us construct the polygon in Figure 4 having the vectors  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  as consecutive sides. The corners of this polygon are labelled  $P$ ,  $Q$ ,  $R$  and  $S$ . It then appears that

$$\begin{aligned} (\mathbf{a} + \mathbf{b}) + \mathbf{c} &= \overrightarrow{PR} + \mathbf{c} \\ &= \overrightarrow{PS}, \end{aligned}$$

$$\begin{aligned} \mathbf{a} + (\mathbf{b} + \mathbf{c}) &= \mathbf{a} + \overrightarrow{QS} \\ &= \overrightarrow{PS}. \end{aligned}$$

Hence the theorem is true.

According to Theorem 2 the sum of three vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  is

independent of the order in which they are added. Hence we can write  $\mathbf{a} + \mathbf{b} + \mathbf{c}$  without ambiguity.

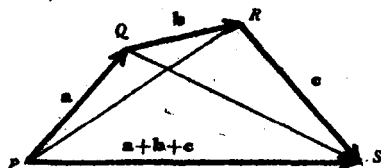


Figure 4

Figure 4 shows the construction of the vector  $\mathbf{a} + \mathbf{b} + \mathbf{c}$ . The sum of a larger number of vectors can be constructed similarly. Thus, to find the vector  $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}$  it is only necessary to construct the polygon having  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  and  $\mathbf{d}$  as consecutive sides. The required vector is then the vector with its origin at the origin of  $\mathbf{a}$ , and its terminus at the terminus of  $\mathbf{d}$ .

3. *Multiplication of a vector by a scalar.* By definition, if  $m$  is a positive scalar and  $\mathbf{a}$  is a vector, the expression  $m\mathbf{a}$  is a vector with magnitude  $m\mathbf{a}$  and pointing in the same direction as  $\mathbf{a}$ ; and if  $m$  is negative,  $m\mathbf{a}$  is a vector with magnitude  $|m|\mathbf{a}$ , and pointing in the direction opposite to  $\mathbf{a}$ .

We note in particular that  $-\mathbf{a}$  is a vector with the same magnitude as  $\mathbf{a}$  but pointing in the direction opposite to  $\mathbf{a}$ . Figure 5 shows this vector, and as further examples of the multiplication of a vector by a scalar, the vectors  $2\mathbf{a}$  and  $-2\mathbf{a}$ .

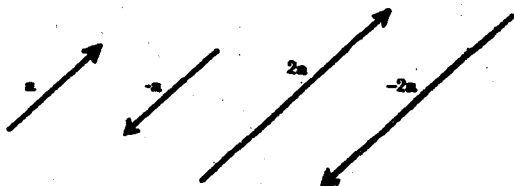


Figure 5

**Theorem.** The multiplication of a vector by a scalar satisfies the distributive laws; that is,

$$(3.1) \quad (m+n)a = ma + na,$$

$$(3.2) \quad m(a+b) = ma + mb.$$

Proof of (3.1). If  $m+n$  is positive, both sides of (3.1) represent a vector with magnitude  $(m+n)a$  and pointing in the same direction as  $a$ . If  $m+n$  is negative, both sides of (3.1) represent a vector with magnitude  $|m+n|a$  and pointing in the direction opposite to  $a$ .

Proof of (3.2). Let  $m$  be positive, and let  $a$ ,  $b$ ,  $ma$  and  $mb$  be as shown in Figures 6 and 7. Then

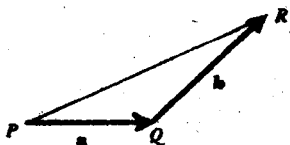


Figure 6

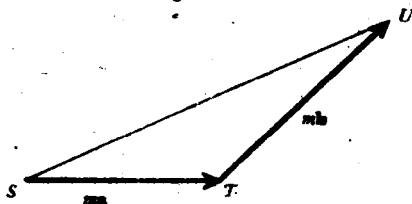


Figure 7

$$(3.3) \quad m(a+b) = m\overline{PR}, \quad ma + mb = \overline{SU}.$$

The two triangles  $PQR$  and  $STU$  are similar. Corresponding sides are then proportional, the constant of proportionality being  $m$ . Thus

$$(3.4) \quad m\overline{PR} = \overline{SU}.$$

Since  $\overline{PR}$  and  $\overline{SU}$  have the same directions, and since  $m$  is positive, then  $m\overline{PR} = \overline{SU}$ . Substitution in both sides of this equation from (3.3) yields (3.2).

Now, let  $m$  be negative. Then Figure 7 is replaced by Figure 8. Equations (3.3) apply in this case also. The triangles  $PQR$  and  $STU$  are again similar, but the constant of proportionality is  $|m|$ , so  $|m|\overline{PR} = \overline{SU}$ . Since  $\overline{PR}$  and  $\overline{SU}$  have opposite directions and  $m$  is negative,

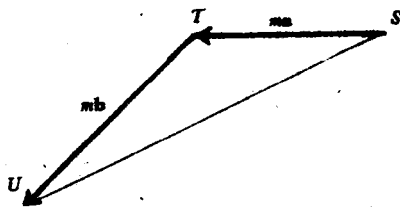


Figure 8

then  $m\overline{PR} = \overline{SU}$ . Substitution in both sides of this equation from (3.3) again yields (3.2).

4. *Subtraction of vectors.* If  $\mathbf{a}$  and  $\mathbf{b}$  are two vectors, their difference  $\mathbf{a} - \mathbf{b}$  is defined by the relation

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}),$$

where the vector  $-\mathbf{b}$  is as defined in the previous section. Figure 9 shows two vectors  $\mathbf{a}$  and  $\mathbf{b}$ , and also their difference  $\mathbf{a} - \mathbf{b}$ .

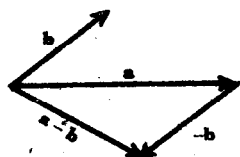


Figure 9

5. *Linear functions.* If  $\mathbf{a}$  and  $\mathbf{b}$  are any two vectors, and  $m$  and  $n$  are any two scalars, the expression  $m\mathbf{a} + n\mathbf{b}$  is called a linear function of  $\mathbf{a}$  and  $\mathbf{b}$ . Similarly,  $m\mathbf{a} + n\mathbf{b} + p\mathbf{c}$  is a linear function of  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$ . The extension of this to the cases involving more than three vectors follows the obvious lines.

*Theorem 1.* If  $\mathbf{a}$  and  $\mathbf{b}$  are any two nonparallel vectors in a plane, and if  $\mathbf{c}$  is any third vector in the plane of  $\mathbf{a}$  and  $\mathbf{b}$ , then  $\mathbf{c}$  can be expressed as a linear function of  $\mathbf{a}$  and  $\mathbf{b}$ .

*Proof.* Since  $\mathbf{a}$  and  $\mathbf{b}$  are not parallel, there exists a parallelogram with  $\mathbf{c}$  as its diagonal and with edges parallel to  $\mathbf{a}$  and  $\mathbf{b}$ . Figure 10 shows this parallelogram. We note from this figure that

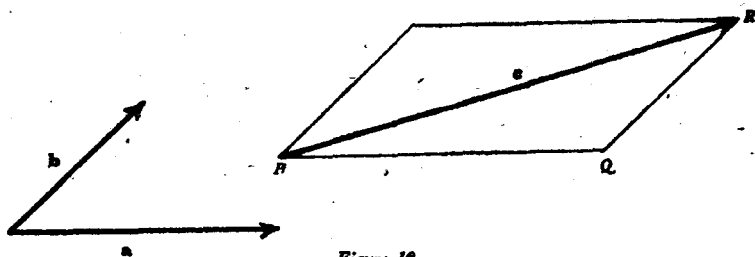


Figure 10

(5.1)

$$\mathbf{c} = \overrightarrow{PQ} + \overrightarrow{QR}.$$

But  $\overrightarrow{PQ}$  is parallel to  $\mathbf{a}$ , and  $\overrightarrow{QR}$  is parallel to  $\mathbf{b}$ . Thus there exist scalars  $m$  and  $n$  such that

$$\overrightarrow{PQ} = m\mathbf{a}, \quad \overrightarrow{QR} = n\mathbf{b}.$$

Substitution from these relations in (5.1) yields

$$\mathbf{c} = m\mathbf{a} + n\mathbf{b}.$$

**Theorem 2.** If  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  are any three vectors not all parallel to a single plane, and if  $\mathbf{d}$  is any other vector, then  $\mathbf{d}$  can be expressed as a linear function of  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$ .

**Proof.** This theorem is the extension of Theorem 1 to space. Since  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  are not parallel to a single plane, there exists a parallelepiped with  $\mathbf{d}$  as its diagonal and with edges parallel to  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$ . Hence there exist scalars  $m$ ,  $n$  and  $p$  such that

$$\mathbf{d} = m\mathbf{a} + n\mathbf{b} + p\mathbf{c}.$$

**6. Rectangular cartesian coordinates.** In much of the theory and application of vectors it is convenient to introduce a set of rectangular cartesian coordinates. We shall *not* denote these by the usual symbols  $x$ ,  $y$  and  $z$ , however, but shall use instead the symbols  $x_1$ ,  $x_2$  and  $x_3$ . These coordinates are said to have "right-handed orientation" or to be "right-handed" if when the thumb of the right hand is made to point in the direction of the positive  $x_3$  axis, the fingers point in the direction of the  $90^\circ$  rotation which carries the positive  $x_1$  axis into coincidence

with the positive  $x_3$  axis. Otherwise the coordinates are "left-handed". In Vector Analysis it is highly desirable to use the same orientation always, for certain basic formulas are changed by a change in orientation. In this book we shall follow the usual practise of using right-handed coordinates throughout. Figure 11 contains the axes of such a set of coordinates.

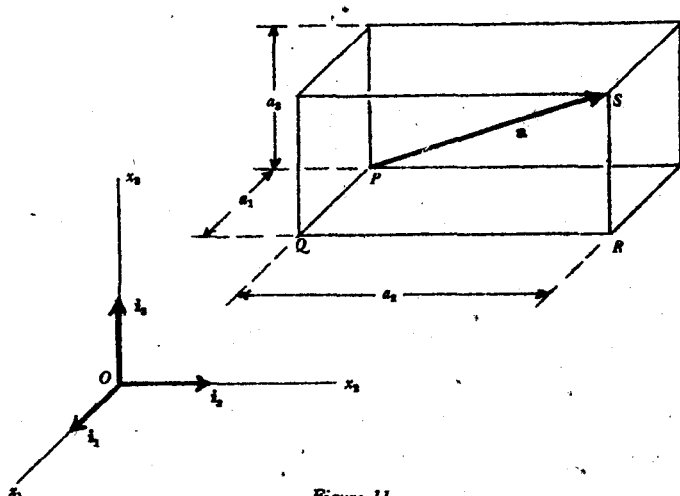


Figure 11

It is also convenient to introduce three vectors of unit magnitude, one pointing in the direction of each of the three positive coordinate axes. These vectors are denoted by  $\mathbf{i}_1$ ,  $\mathbf{i}_2$  and  $\mathbf{i}_3$ , and are shown in Figure 11.

Let us consider a vector  $\mathbf{a}$ . It has orthogonal projections in the directions of the positive coordinate axes. These are denoted by  $a_1$ ,  $a_2$  and  $a_3$ , as shown in Figure 11. They are called the components of  $\mathbf{a}$ . It should be noted that they can be positive or negative. Thus, for example,  $a_1$  is positive when the angle between  $\mathbf{a}$  and the direction of the positive  $x_1$  axis (the angle  $QPS$  in the figure) is acute, and is negative when this angle is obtuse.

From Figure 11 it also appears that  $\mathbf{a}$  is the diagonal of a rectangular

parallelepiped whose edges have lengths  $|a_1|$ ,  $|a_2|$  and  $|a_3|$ . Hence the magnitude  $a$  of the vector  $\mathbf{a}$  is given by the relation

$$(6.1) \quad a = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

From the figure it also appears that

$$(6.2) \quad \mathbf{a} = \overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RS}.$$

Now the vector  $\overrightarrow{PQ}$  is parallel to  $\mathbf{i}_1$ . Because of the definitions of  $a_1$  and of the product of a scalar by a vector, we then have the relation  $\overrightarrow{PQ} = a_1 \mathbf{i}_1$ . Similarly  $\overrightarrow{QR} = a_2 \mathbf{i}_2$  and  $\overrightarrow{RS} = a_3 \mathbf{i}_3$ . Substitution in (6.2) from these relations yields

$$(6.3) \quad \mathbf{a} = a_1 \mathbf{i}_1 + a_2 \mathbf{i}_2 + a_3 \mathbf{i}_3.$$

This relation expresses the vector  $\mathbf{a}$  as a linear function of the unit vectors  $\mathbf{i}_1$ ,  $\mathbf{i}_2$  and  $\mathbf{i}_3$ . We note that the coefficients are the components of  $\mathbf{a}$ .

*Theorem.* The components of the sum of a number of vectors are equal to the sums of the components of the vectors.

*Proof.* We consider two vectors  $\mathbf{a}$  and  $\mathbf{b}$  with components  $a_1, a_2, a_3, b_1, b_2$  and  $b_3$ . Then

$$\begin{aligned} \mathbf{a} &= a_1 \mathbf{i}_1 + a_2 \mathbf{i}_2 + a_3 \mathbf{i}_3, \\ \mathbf{b} &= b_1 \mathbf{i}_1 + b_2 \mathbf{i}_2 + b_3 \mathbf{i}_3. \end{aligned}$$

Addition of both sides of these equations leads to the relation

$$\mathbf{a} + \mathbf{b} = a_1 \mathbf{i}_1 + a_2 \mathbf{i}_2 + a_3 \mathbf{i}_3 + b_1 \mathbf{i}_1 + b_2 \mathbf{i}_2 + b_3 \mathbf{i}_3.$$

Now the sum of a number of vectors is independent of the order in which the vectors are added, by Theorem 1 of § 2. Hence we may write the above equation in the form

$$\mathbf{a} + \mathbf{b} = a_1 \mathbf{i}_1 + b_1 \mathbf{i}_1 + a_2 \mathbf{i}_2 + b_2 \mathbf{i}_2 + a_3 \mathbf{i}_3 + b_3 \mathbf{i}_3.$$

By the theorem in § 3 we may then write this in the form

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1) \mathbf{i}_1 + (a_2 + b_2) \mathbf{i}_2 + (a_3 + b_3) \mathbf{i}_3.$$

Hence the components of  $\mathbf{a} + \mathbf{b}$  are  $a_1 + b_1$ ,  $a_2 + b_2$  and  $a_3 + b_3$ . This proves the theorem when two vectors are added. The proof is similar when more than two vectors are added.



7. *The scalar product.* Let us consider two vectors  $\mathbf{a}$  and  $\mathbf{b}$  with magnitudes  $a$  and  $b$ , respectively. Let  $\alpha$  be the smallest nonnegative angle between  $\mathbf{a}$  and  $\mathbf{b}$ , as shown in Figure 12. Then  $0^\circ < \alpha < 180^\circ$ .

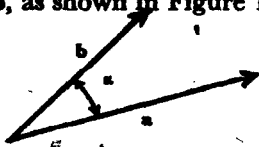


Figure 12

The scalar  $ab \cos \alpha$  arises quite frequently, and hence it is convenient to give it a name. It is called the scalar product of  $\mathbf{a}$  and  $\mathbf{b}$ . It is also denoted by the symbols  $\mathbf{a} \cdot \mathbf{b}$ , and hence we have

$$(7.1) \quad \mathbf{a} \cdot \mathbf{b} = ab \cos \alpha.$$

The scalar product is sometimes referred to as the dot product.

If the components of  $\mathbf{a}$  and  $\mathbf{b}$  are denoted by  $a_1, a_2, a_3, b_1, b_2, b_3$  in the usual manner, the direction cosines of the directions of  $\mathbf{a}$  and  $\mathbf{b}$  are respectively

$$\frac{a_1}{a}, \frac{a_2}{a}, \frac{a_3}{a}; \quad \frac{b_1}{b}, \frac{b_2}{b}, \frac{b_3}{b}.$$

By a formula of analytic geometry, we then have

$$\cos \alpha = \frac{a_1}{a} \frac{b_1}{b} + \frac{a_2}{a} \frac{b_2}{b} + \frac{a_3}{a} \frac{b_3}{b}.$$

Substitution in (7.1) of this expression for  $\cos \alpha$  yields

$$(7.2) \quad \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

This relation expresses the scalar product of two vectors in terms of the components of the vectors.

*Theorem 1.* The scalar product is commutative; that is,

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}.$$

*Proof.* Because of (7.2), we have

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3,$$

$$\mathbf{b} \cdot \mathbf{a} = b_1 a_1 + b_2 a_2 + b_3 a_3.$$