VECTOR & TENSOR ANALYSIS

G. E. HAY

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by

G. E. HAY

Associate Professor of Mathematics
University of Michigan

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TABLE OF CONTENTS

CHAPTER I

ELEMENTARY OPERATIONS

1. *Definitions.* Quantities which have magnitude only are called *scalars.* The following are examples: mass, distance, area, volume. A scalar can be represented by a number with an associated sign, which indicates its magnitude to some convenient scale.

There are quantities which have not only magnitude but also direction. The following are examples: force, displacement of a point, velocity of a point, acceleration of a point. Such quantities are called *vectors* if they obey a certain law of addition set forth in § 2 below. A vector can be represented by an arrow. The direction of the arrow indicates the direction of the vector, and the length of the arrow indicates the magnitude of the vector to some convenient scale.

Let us consider a vector represented by an arrow running from a point P to a point Q, as shown in Figure 1. The straight line through P and Q is called the *line of action* of the vector, the point P is called the *origin* of the vector, and the point Q is called the *terminus* of the vector.

To denote a vector we write the letter indicating its origin followed by the letter indicating its terminus, and place a bar over the two letters. The vector represented in Figure 1 is then represented by the symbols $\overline{PQ}$. In this book the superimposed bar will not be used in any capacity other than the above, and hence its presence can always

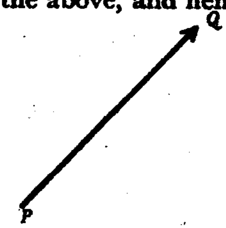

Figure 1

be interpreted as denoting vector character. This notation for vectors is somewhat cumbersome. Hence when convenient we shall use a simpler notation which consists in denoting a vector by a single symbol in bold-faced type. Thus, the vector in Figure 1 might be denoted by the symbol **a**. In this book no mathematical symbols will be printed in bold-faced type except those denoting vectors.*

The magnitude of a vector is a scalar which is never negative. The magnitude of a vector $\overline{PQ}$ will be denoted by either PQ or $|\overline{PQ}|$. Similarly, the magnitude of a vector **a** will be denoted by either a or $|\mathbf{a}|$.

Two vectors are said to be equal if they have the same magnitudes and the same directions. To denote the equality of two vectors the usual sign is employed. Hence, if **a** and **b** are equal vectors, we write

$$\mathbf{a} = \mathbf{b}.$$

A vector **a** is said to be equal to zero if its magnitude a is equal to zero. Thus $\mathbf{a} = 0$ if $a = 0$. Such a vector is called a zero vector.

2. *Addition of vectors.* In § 1 it was stated that vectors are quantities with magnitude and direction, and which obey a certain law of addition. This law, which is called the *law of vector addition*, is as follows.

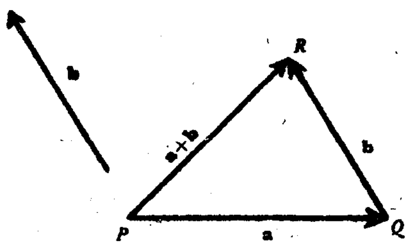

Figure 2

Let **a** and **b** be two vectors, as shown in Figure 2. The origin and terminus of **a** are P and Q. A vector equal to **b** is constructed with

* It is difficult to write bold-faced symbols on the blackboard or in the exercise book. When it is desired to write a single symbol denoting a vector, the reader will find it convenient to write the symbol in the ordinary manner, and to place a bar over it to indicate vector character.

its origin at Q. Its terminus falls at a point R. The sum $\mathbf{a}+\mathbf{b}$ is the vector $\overline{PR}$, and we write

$$\mathbf{a}+\mathbf{b} = \overline{PR}.$$

Theorem 1. Vectors satisfy the commutative law of addition; that is,

$$\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}.$$

Proof. Let $\mathbf{a}$ and $\mathbf{b}$ be the two vectors shown in Figure 2. Then

$$\mathbf{a}+\mathbf{b} = \overline{PR}. \tag{2.1}$$

We now construct a vector equal to $\mathbf{b}$, with its origin at P. Its terminus falls at a point S. A vector equal to $\mathbf{a}$ is then constructed with its origin at S. The terminus of this vector will fall at R, and Figure 3 results. Hence

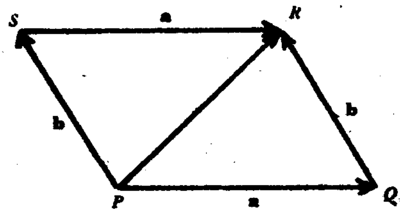

Figure 3

$$\mathbf{b}+\mathbf{a} = \overline{PR}. \tag{2.2}$$

From (2.1) and (2.2) it follows that $\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$.

Theorem 2. Vectors satisfy the associative law of addition; that is,

$$(\mathbf{a}+\mathbf{b})+\mathbf{c} = \mathbf{a}+(\mathbf{b}+\mathbf{c}).$$

Proof. Let us construct the polygon in Figure 4 having the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ as consecutive sides. The corners of this polygon are labelled P, Q, R and S. It then appears that

$$\begin{aligned}(\mathbf{a}+\mathbf{b})+\mathbf{c} &= \overline{PR}+\mathbf{c} \\ &= \overline{PS}, \\ \mathbf{a}+(\mathbf{b}+\mathbf{c}) &= \mathbf{a}+\overline{QS} \\ &= \overline{PS}.\end{aligned}$$

Hence the theorem is true.

According to Theorem 2 the sum of three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is independent of the order in which they are added. Hence we can write $\mathbf{a}+\mathbf{b}+\mathbf{c}$ without ambiguity.

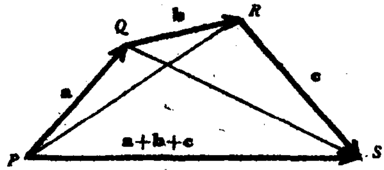

Figure 4

Figure 4 shows the construction of the vector $\mathbf{a}+\mathbf{b}+\mathbf{c}$. The sum of a larger number of vectors can be constructed similarly. Thus, to find the vector $\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}$ it is only necessary to construct the polygon having $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ as consecutive sides. The required vector is then the vector with its origin at the origin of $\mathbf{a}$, and its terminus at the terminus of $\mathbf{d}$.

3. *Multiplication of a vector by a scalar.* By definition, if m is a positive scalar and $\mathbf{a}$ is a vector, the expression $m\mathbf{a}$ is a vector with magnitude ma and pointing in the same direction as $\mathbf{a}$; and if m is negative, $m\mathbf{a}$ is a vector with magnitude $|m|\,a$, and pointing in the direction opposite to $\mathbf{a}$.

We note in particular that $-\mathbf{a}$ is a vector with the same magnitude as $\mathbf{a}$ but pointing in the direction opposite to $\mathbf{a}$. Figure 5 shows this vector, and as further examples of the multiplication of a vector by a scalar, the vectors $2\mathbf{a}$ and $-2\mathbf{a}$.

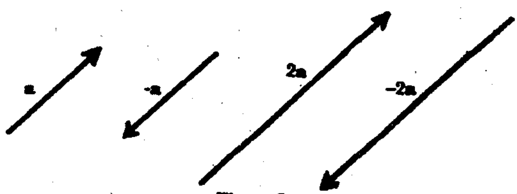

Figure 5

Theorem. The multiplication of a vector by a scalar satisfies the distributive laws; that is, (3.1) $$(m+n)\mathbf{a} = m\mathbf{a}+n\mathbf{a},$$

(3.2) $$m(\mathbf{a}+\mathbf{b}) = m\mathbf{a}+m\mathbf{b}.$$

Proof of (3.1). If $m+n$ is positive, both sides of (3.1) represent a vector with magnitude $(m+n)a$ and pointing in the same direction as $\mathbf{a}$. If $m+n$ is negative, both sides of (3.1) represent a vector with magnitude $|m+n|a$ and pointing in the direction opposite to $\mathbf{a}$.

Proof of (3.2). Let m be positive, and let $\mathbf{a}$, $\mathbf{b}$, $m\mathbf{a}$ and $m\mathbf{b}$ be as shown in Figures 6 and 7. Then

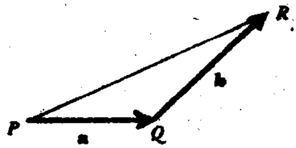

Figure 6

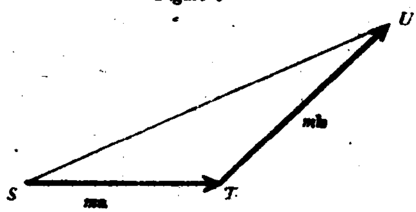

Figure 7

(3.3) $$m(\mathbf{a}+\mathbf{b}) = m\overline{PR}, \quad m\mathbf{a}+m\mathbf{b} = \overline{SU}.$$

The two triangles PQR and STU are similar. Corresponding sides are then proportional, the constant of proportionality being m. Thus

(3.4) $$mPR=SU.$$

Since $\overline{PR}$ and $\overline{SU}$ have the same directions, and since m is positive, then $m\overline{PR} = \overline{SU}$. Substitution in both sides of this equation from (3.3) yields (3.2).

Now, let m be negative. Then Figure 7 is replaced by Figure 8. Equations (3.3) apply in this case also. The triangles PQR and STU are again similar, but the constant of proportionality is $|m|$, so $|m|PR = SU$. Since $\overline{PR}$ and $\overline{SU}$ have opposite directions and m is negative,

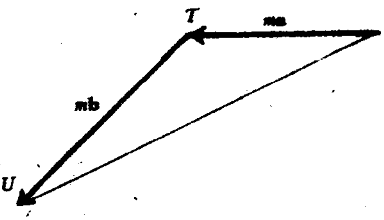

Figure 8

then $m\overline{PR} = \overline{SU}$. Substitution in both sides of this equation from (3.3) again yields (3.2).

4. *Subtraction of vectors.* If **a** and **b** are two vectors, their difference **a** − **b** is defined by the relation

$$\mathbf{a}-\mathbf{b} = \mathbf{a}+(-\mathbf{b}),$$

where the vector −**b** is as defined in the previous section. Figure 9 shows two vectors **a** and **b**, and also their difference **a** − **b**.

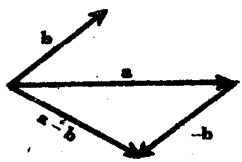

Figure 9

5. *Linear functions.* If **a** and **b** are any two vectors, and m and n are any two scalars, the expression $m\mathbf{a}+n\mathbf{b}$ is called a linear function of **a** and **b**. Similarly, $m\mathbf{a}+n\mathbf{b}+p\mathbf{c}$ is a linear function of **a**, **b**, and **c**. The extension of this to the cases involving more than three vectors follows the obvious lines.

Theorem 1. If **a** and **b** are any two nonparallel vectors in a plane, and if **c** is any third vector in the plane of **a** and **b**, then **c** can be expressed as a linear function of **a** and **b**.

Proof. Since **a** and **b** are not parallel, there exists a parallelogram with **c** as its diagonal and with edges parallel to **a** and **b**. Figure 10 shows this parallelogram. We note from this figure that

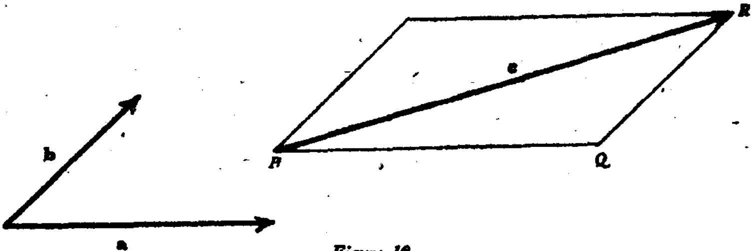

Figure 10

$$\mathbf{c} = \overline{PQ} + \overline{QR}. \tag{5.1}$$

But $\overline{PQ}$ is parallel to **a**, and $\overline{QR}$ is parallel to **b**. Thus there exist scalars m and n such that

$$\overline{PQ} = m\mathbf{a}, \qquad \overline{QR} = n\mathbf{b}.$$

Substitution from these relations in (5.1) yields

$$\mathbf{c} = m\mathbf{a} + n\mathbf{b}.$$

Theorem 2. If **a**, **b** and **c** are any three vectors not all parallel to a single plane, and if **d** is any other vector, then **d** can be expressed as a linear function of **a**, **b** and **c**.

Proof. This theorem is the extension of Theorem 1 to space. Since **a**, **b** and **c** are not parallel to a single plane, there exists a parallelepiped with **d** as its diagonal and with edges parallel to **a**, **b** and **c**. Hence there exist scalars m, n and p such that

$$\mathbf{d} = m\mathbf{a} + n\mathbf{b} + p\mathbf{c}.$$

6. *Rectangular cartesian coordinates*. In much of the theory and application of vectors it is convenient to introduce a set of rectangular cartesian coordinates. We shall *not* denote these by the usual symbols x, y and z, however, but shall use instead the symbols x_1, x_2 and x_3. These coordinates are said to have "right-handed orientation" or to be "right-handed" if when the thumb of the right hand is made to point in the direction of the positive x_3 axis, the fingers point in the direction of the 90° rotation which carries the positive x_1 axis into coincidence

with the positive x_3 axis. Otherwise the coordinates are "left-handed". In Vector Analysis it is highly desirable to use the same orientation always, for certain basic formulas are changed by a change in orientation. *In this book we shall follow the usual practise of using right-handed coordinates throughout.* Figure 11 contains the axes of such a set of coordinates.

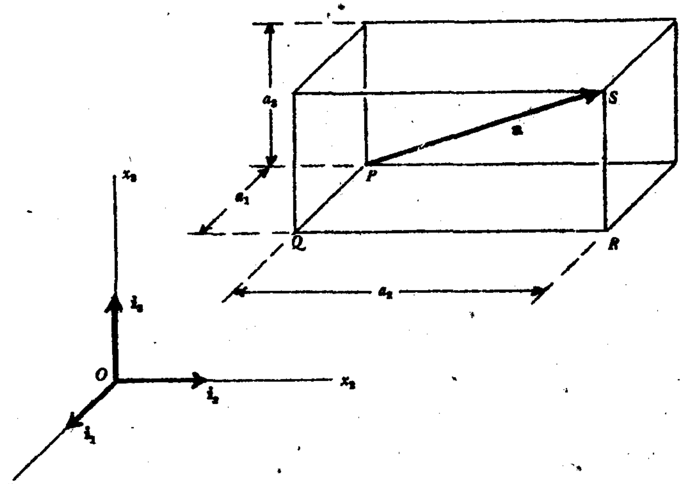

Figure 11

It is also convenient to introduce three vectors of unit magnitude, one pointing in the direction of each of the three positive coordinate axes. These vectors are denoted by $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$, and are shown in Figure 11.

Let us consider a vector $\mathbf{a}$. It has orthogonal projections in the directions of the positive coordinate axes. These are denoted by a_1, a_2 and a_3, as shown in Figure 11. They are called the components of $\mathbf{a}$. It should be noted that they can be positive or negative. Thus, for example, a_1 is positive when the angle between $\mathbf{a}$ and the direction of the positive x_1 axis (the angle QPS in the figure) is acute, and is negative when this angle is obtuse.

From Figure 11 it also appears that $\mathbf{a}$ is the diagonal of a rectangular parallelepiped whose edges have lengths $|a_1|$, $|a_2|$ and $|a_3|$. Hence the magnitude a of the vector $\mathbf{a}$ is given by the relation

$$(6.1) \qquad a = \sqrt{a_1^2+a_2^2+a_3^2}.$$

From the figure it also appears that

$$(6.2) \qquad \mathbf{a} = \overline{PQ}+\overline{QR}+\overline{RS}.$$

Now the vector $\overline{PQ}$ is parallel to $\mathbf{i}_1$. Because of the definitions of a_1 and of the product of a scalar by a vector, we then have the relation $\overline{PQ} = a_1\mathbf{i}_1$. Similarly $\overline{QR} = a_2\mathbf{i}_2$ and $\overline{RS} = a_3\mathbf{i}_3$. Substitution in (6.2) from these relations yields

$$(6.3) \qquad \mathbf{a} = a_1\mathbf{i}_1+a_2\mathbf{i}_2+a_3\mathbf{i}_3.$$

This relation expresses the vector $\mathbf{a}$ as a linear function of the unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$. We note that the coefficients are the components of $\mathbf{a}$.

Theorem. The components of the sum of a number of vectors are equal to the sums of the components of the vectors.

Proof. We consider two vectors $\mathbf{a}$ and $\mathbf{b}$ with components a_1, a_2, a_3, b_1, b_2 and b_3. Then

$$\mathbf{a} = a_1\mathbf{i}_1+a_2\mathbf{i}_2+a_3\mathbf{i}_3,$$
$$\mathbf{b} = b_1\mathbf{i}_1+b_2\mathbf{i}_2+b_3\mathbf{i}_3.$$

Addition of both sides of these equations leads to the relation

$$\mathbf{a}+\mathbf{b} = a_1\mathbf{i}_1+a_2\mathbf{i}_2+a_3\mathbf{i}_3+b_1\mathbf{i}_1+b_2\mathbf{i}_2+b_3\mathbf{i}_3.$$

Now the sum of a number of vectors is independent of the order in which the vectors are added, by Theorem 1 of § 2. Hence we may write the above equation in the form

$$\mathbf{a}+\mathbf{b} = a_1\mathbf{i}_1+b_1\mathbf{i}_1+a_2\mathbf{i}_2+b_2\mathbf{i}_2+a_3\mathbf{i}_3+b_3\mathbf{i}_3.$$

By the theorem in § 3 we may then write this in the form

$$\mathbf{a}+\mathbf{b} = (a_1+b_1)\mathbf{i}_1+(a_2+b_2)\mathbf{i}_2+(a_3+b_3)\mathbf{i}_3.$$

Hence the components of $\mathbf{a}+\mathbf{b}$ are a_1+b_1, a_2+b_2 and a_3+b_3. This proves the theorem when two vectors are added. The proof is similar when more than two vectors are added.

7. *The scalar product.* Let us consider two vectors **a** and **b** with magnitudes a and b, respectively. Let α be the smallest nonnegative angle between **a** and **b**, as shown in Figure 12. Then $0^\circ < \alpha < 180^\circ$.

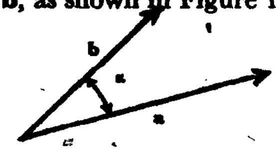

Figure 12

The scalar $ab \cos \alpha$ arises quite frequently, and hence it is convenient to give it a name. It is called the scalar product of **a** and **b**. It is also denoted by the symbols $\mathbf{a} \cdot \mathbf{b}$, and hence we have

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \alpha. \tag{7.1}$$

The scalar product is sometimes referred to as the dot product.

If the components of **a** and **b** are denoted by a_1, a_2, a_3, b_1, b_2 and b_3 in the usual manner, the direction cosines of the directions of **a** and **b** are respectively

$$\frac{a_1}{a}, \ \frac{a_2}{a}, \ \frac{a_3}{a}; \quad \frac{b_1}{b}, \ \frac{b_2}{b}, \ \frac{b_3}{b}.$$

By a formula of analytic geometry, we then have

$$\cos \alpha = \frac{a_1}{a} \frac{b_1}{b} + \frac{a_2}{a} \frac{b_2}{b} + \frac{a_3}{a} \frac{b_3}{b}.$$

Substitution in (7.1) of this expression for $\cos \alpha$ yields

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3. \tag{7.2}$$

This relation expresses the scalar product of two vectors in terms of the components of the vectors.

Theorem 1. The scalar product is commutative; that is,

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}.$$

Proof. Because of (7.2), we have

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3,$$
$$\mathbf{b} \cdot \mathbf{a} = b_1 a_1 + b_2 a_2 + b_3 a_3.$$