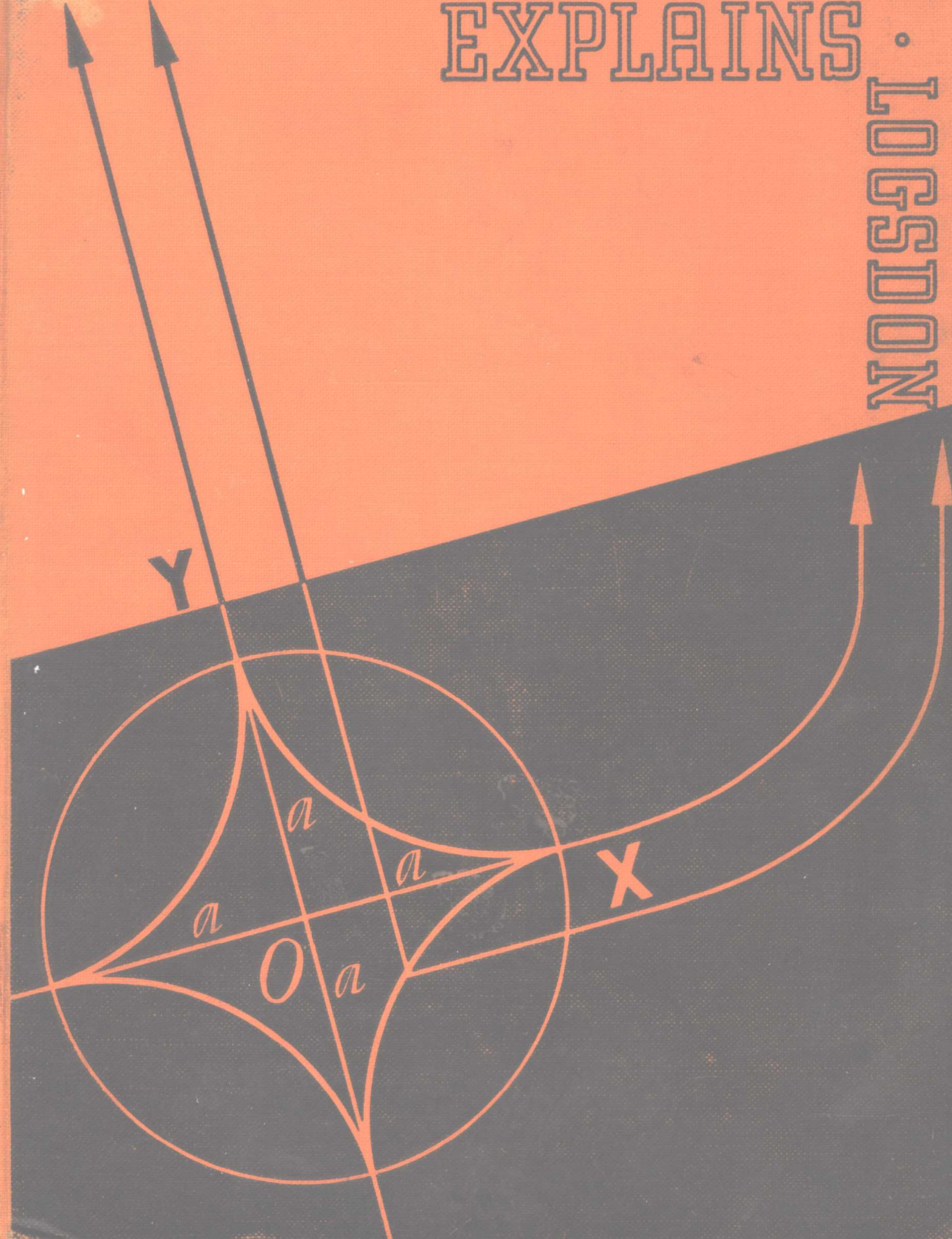
A MATHEMATICIAN EXPLAINS

LOGSDON



A MATHEMATICIAN EXPLAINS

BY

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DECORATIVE DRAWINGS BY

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THE UNIVERSITY OF CHICAGO PRESS

CHICAGO · ILLINOIS

PREFACE

THE idea that each individual has something to gain by acquiring a knowledge of mathematics is not new. According to Plato,* "the art of calculation (*logistika*) and arithmetic are both concerned with number; those who have a natural gift for calculating have, generally speaking, a talent for learning of all kinds, and even those who are slow are, by practice in it, made smarter. But the art of calculation is only preparatory to the true science; those who are to govern the city are to get a grasp of *logistika*, not in the popular sense with a view to use in trade, but only for the purpose of knowledge, until they are able to contemplate the nature of number in itself by thought alone."

The college curriculum of the University of Chicago, adopted in 1930, includes a course of three lectures per week for one year in the physical sciences, paralleled with small-group discussions. The conduct and content of this course, which is required of all college students, are motivated by the knowledge that many persons pass through youth, adulthood, and old age with no understanding of, or interest in, the operation of natural laws.

This course, planned for the student who has no native interest in the physical sciences, is perforce entirely different from a course which might be designed for a student who has a definite interest in this field. Its aim is to provide explanations for what is happening about us and to show with some detail how the human race through the ages has arrived

*Adapted from Plato, *The Republic*, pp. 525–26.

at the explanations here presented. In this development mathematics plays a twofold rôle, viz., the unfolding and attaining of pure mathematical theory and the invention of mathematical processes as aids to astronomy, physics, geology, and chemistry. In ancient times the latter was the predominating function of the science called "exact"; in modern times the former aspect of mathematics has become of ever increasing importance.

The attempt to give in a few lectures a vivid picture of the historical development of the mathematics of classical times with a description of the types of problems which led to the growth of elementary concepts of arithmetic, algebra, geometry, and trigonometry, and to give something of the purport and processes of the modern subjects, analytical geometry and the calculus, to the end that the student may obtain fairly definite ideas of their meanings and uses in modern life and of their relations to the various fields of the physical sciences, has been rendered more difficult than pleasant by the lack of satisfactory references for extensive reading; and it is to meet that need that this book has been written. In it the subjects which may be considered important for the general education of a person who is not a specialist in a physical science have a more complete treatment than can be given in a few lectures, but at the same time the text does not go so far afield as to confuse with new ideas or with technical notions. It does not take the place of any one or more texts in the standard courses in college mathematics, but its sponsors believe that it will prove to be of use along the following lines:

(1) To provide the mathematics for general physical science courses, as at the University of Chicago.

(2) To serve as a text for a one-hour or a two-hour orientation course in college, junior college, or senior high school.

(3) To serve as a reading reference for first-year and second-year mathematical courses in college or junior college.

(4) To serve as a supplementary text for courses in the teaching of mathematics in normal schools and teachers colleges.

(5) To serve as an eye-opener for the adult who knows no mathematics beyond elementary algebra and geometry but who has a healthy curiosity concerning the science whose development has made possible this age of the machine.

Acknowledgments are due and are gratefully rendered to Professor Gilbert A. Bliss, who contributed chapter 8, "Mathematical Interpretations of Geometrical and Physical Phenomena" (two lectures in the general course mentioned above); to the editors of the American Mathematical Monthly, for permission to reprint it (with minor changes) from Vol. 40 (1933); to Mrs. Ardis Monk, who read the manuscript and whose experience as a discussion group leader enabled her to offer valuable suggestions as to emphasis and method of presentation; to Mr. Carl Denbow, Fellow in mathematics at the University of Chicago, whose constructive criticism of the manuscript resulted in the smoothing-out of many rough places; to Mr. A. Boyd Newborn of the University of Arizona for aid in preparing copy for the cuts; to Mrs. Chichi Lasley, the artist who provided the illustrations; and to Mr. Donald Bean and his able assistants who saw the book through the press.

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THE UNIVERSITY OF CHICAGO
October 3, 1935

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CHAPTER 1

NATURE OF MATHEMATICS

A PUPIL of Euclid, when he had learned a proposition, inquired: "What advantage shall I get by learning these things?" Euclid called a slave and said, "Give him a sixpence, since he must needs gain by what he learns."

It is not the purpose of this book to attempt to give to the reader a knowledge and skill in the use of mathematics which will make of him a better money-gatherer, but rather to come to the aid of the many who reach adult years with a distaste for mathematics and a pronounced inferiority complex with regard to it, and who at the same time suffer an occasional feeling of embarrassment at their inability to understand some apparently simple natural or mechanical law. Most of these persons have more or less curiosity concerning the science of which they know little and would gladly undertake to acquire an understanding of the origins and uses of mathematics if they were convinced that this could be done in an informal way without the necessity of drawing on a supposed-known-but-long-since-forgotten secondary-school mathematical training.

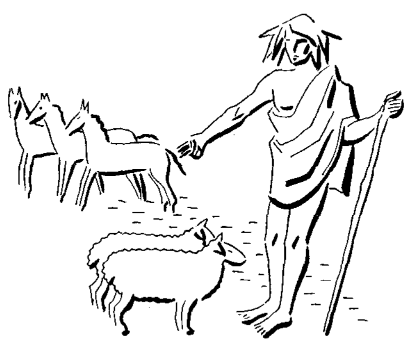
Elementary mathematics includes arithmetic, algebra, geometry, trigonometry, analytical geometry, and the calculus. We shall try to trace some of the important steps in the development of these subjects from their beginnings, showing that an advance in the growth of the science was the direct or indirect attempt to *satisfy a definite human need.*

A second objective is to show something of the importance of mathematics and the mathematical sciences in *enriching the intellectual life* of the twentieth century, as well as in contributing to the *physical comfort* and the *recreational pleasures*.

A third objective is to look carefully into the *nature of the science* which is commonly labeled "abstract" and "deductive," and show that these descriptive terms need not imply that behind them lie mystery and difficulty of comprehension, but rather beauty, elegance, and, above all, *orderliness* and *simplicity*.

The history of mathematics dates back to the beginning of civilization, and there is a remarkable parallelism between the various stages of its development and the mathematical experiences of each reader of these pages through his childhood, his youth, and his adult years. By constantly keeping this parallelism in evidence, by giving numerous specific examples and outlining their methods of attack, by keeping the language as nontechnical as is consistent with accuracy and clear thinking, and by showing many connections of mathematics with everyday life, the author hopes to give to the general reader a picture of what mathematics is, what it does, and (a very little of) how it does it.

Indeed, a little attention leads to the feeling that this science, though rightly called "abstract," is, in reality, deeply human and alive and that it is not impossible to inform one's self concerning the why's and how's without actually acquiring the technique of the doing; for, let it be definitely understood that this book is a book of information, not a teaching book. In the presentation of scientific facts, however, it is inevitable that enough of method must be given to enable the reader to follow a succession of steps to a logical conclusion. Hence it is hoped that an important by-product of the reading of this book will be the intellectual appreciation of the power and the elegance of deductive reasoning.



1. Mathematics an abstract science. It is certain that primitive man, of whatever race, had a manner of designating two sheep, three horses, ten warriors, long before he had a concept of, and words or symbols for, the abstract numbers 2, 3, 10. The small child has an understanding of what is meant by two apples, two chairs, five marbles, before he is able to abstract the numeral from its noun. Early problems in the kindergarten and elementary grades are concrete:

Tom has 10 cents and spends 2 cents for a pencil. How many cents has he left?

John has three apples and his brother has three apples. How many apples do the two boys have?

How much will five books cost at 50 cents each?

A little thought convinces that the intellectual processes involved in solving these "story" problems are simpler in the beginning than those needed for the comprehension and solution of

$$10-2=? \qquad 3+3=? \qquad 5\times 50=?$$

When we contemplate the nature of mathematics, we are struck by very significant facts. For example, $3+4=7$ may be thought of as relating to dollars or leaves or stars or what you will, but the sentence $3+4=7$ is the statement of a mathematical fact which is not necessarily thought of as associated with any physical object or even idea. Indeed, as the study of arithmetic progresses, there seems to be a definite and conscious effort to detach the arithmetical operations from human experiences. The relative number of concrete problems, as compared with the number of abstract computational problems, decreases. It is not surprising that many young people early acquire the pronounced feeling that there is no especial human significance in the science of mathematics.

Let us see if, by looking at particular instances, we can comprehend the reason for the apparent divorcing of mathematics from human experience. We all know more or less precisely what the Law of Gravitation is. We understand that this law explains many phenomena of our everyday life. For example, an apple falls to the earth, the moon does not fall to the earth but revolves around the earth, water runs down hill, a ball when thrown into the air falls back to the earth, the moon produces tides on the earth's surface—these are facts of observation explained by the Law of Gravitation. But the mathematical explanation of these phenomena makes no mention of the apple, ball, moon, etc. It may be written

$$f=k\frac{mM}{d^2}, \qquad (1)$$

where m and M are the masses of two bodies, d is the distance between them,* and k is a constant depending on the units of measure, of mass, and of distance. The force f gives the magnitude of the attraction exerted by each body upon the other. Its direction is along the line which joins their centers of gravity.

In equation (1) we have a statement which, assuming an understanding of the laws of motion, not only explains the motion of the apple, the ball, and the other objects mentioned above and of all other earthly objects which have moved, are moving, or will in the future move subject only to the attraction of gravitation, but which also explains the motions of the planets around the sun, of the stars in their courses, of bodies in this universe or in any hypothetical universe in which are being considered the motions of bodies which are subject to no force other than that of the attraction of gravitation.

Newton could never have obtained his Law of Gravitation if he had not noted *that which is common* to a great number of apparently unrelated natural phenomena; and his statement of this law, in which, by ignoring specific examples, he appears to remove it from the realm of everyday experience, actually leaves it free for an unlimited scope of application. *Only by an abstract statement can the field of application be completely unrestricted.*

It is neither possible nor desirable here to attempt to show how the foregoing formula (it is called the *Law of Inverse Squares*) solves all of the problems mentioned, but in the next paragraphs we shall look more closely at the problem of the falling apple. The apple falls to the ground because of the attraction of the earth for it. As long as the stem is green and strong, it can resist the pull of the earth on the apple, although that pull is by no means a slight pull; but when the apple is ripe, the stem becomes more brittle and there comes a time when the pull of the earth (the attraction of gravitation) is strong enough to break the stem, and the apple falls. It is inevitable that there should be motion when a force is acting unless the force is neutralized by a second force. The magnitude of the force, as we learn from equation (1), depends on the mass of the earth, the mass of the apple, and the distance between them; the amount of motion depends on the magnitude of the force.

* By "distance between them" is meant distance between their centers of gravity.

Experiments are frequently performed in physics laboratories to demonstrate the action of bodies free to move when acted upon by an exterior force. Before the time of GALILEO (1564–1642) neither the mathematicians nor the philosophers could agree on whether distances traversed by objects which were subjected to the same force depend on the action of the force alone or on the time during which the force is acting, or on both, or whether the masses of the objects are also effective in determining distances. This is not surprising; in fact, this question could not be answered with precision until Galileo demonstrated that the size and mass of an object falling freely or sliding (with negligible friction) down an inclined plane have no effect on its velocity. Further experiments then and later show that

a) *The distance traversed the second second is three times that of the first second,*
b) *The distance traversed the third second is five times that of the first second,*
c) *The distance traversed the fourth second is seven times that of the first second,*
d) *The distance traversed the fifth second is nine times that of the first second, etc.*

It is merely repeating the information above, but from a slightly different point of view, if we say:

a) *The distance traversed during the first 2 sec. is four times that of the first second,*
b) *The distance traversed during the first 3 sec. is nine times that of the first second,*
c) *The distance traversed during the first 4 sec. is sixteen times that of the first second,*
d) *The distance traversed the first 5 sec. is twenty-five times that of the first second, etc.*

Now, a mathematician examines four statements like the last four and seeks for something common in them which he can express in a formula—a formula which not only tells all that has been told in the four statements but also tells what is concealed in the "etc." Without a formula a reader would conclude that the experiments have been carried out and data obtained for a longer interval than 5 sec. but that it is not necessary to take up more space in writing it out, since further items are exactly what one would expect them to be in the light of what has been displayed.

In order to tell all of this and more in a formula, obviously a number symbol must be used to designate the number of seconds, a second symbol is needed to denote the distance traversed during the first second, and a third for the total distance from the beginning. Let us use t for the number of seconds, d for the distance during the first second, and s for the total distance. Let us then write the formula,

$$s = dt^2 ,$$

and examine it to see if it gives all of the information which we expect of it. First, put $t=1$ and you see that $s=d$; put $t=2$ and compute $s=4d$; put $t=3$ and get $s=9d$; it really is not necessary to go further except to say that, if the data are correct and if we are really justified in believing that the distance continues to behave even for large values of t as it behaves for t not greater than 5, we can use the formula to compute the distance traversed in the first 10 sec., in the first 30 sec., in any chosen number of seconds without writing down the intermediate distances for 6, 7, 8, 9, etc., sec., as would be necessary if we worked only from the information in the form in which it was first given.

As a matter of fact, experience has shown that the formula

$$s = dt^2$$

is correct for an object moving without friction near the surface of the earth when there is no changing force acting on it except the attraction of the earth. Hence, if we know the distance which a given object has moved during the first second, its distance for any desired number of seconds can be computed.

Now let us look into the possible values of d. Again we go to the laboratory and devise the following experiment: A number of wires are set up as in the drawing with varying slants or slopes. On each wire is a bead, and at the beginning of the experiment the beads are equally distant from the floor. They are automatically released at a given instant, and pictures are taken with a motion-picture camera until the first bead reaches the floor. The pictures show that the greatest actual distance (along its wire), as well as the greatest vertical distance, is traversed by the bead on the vertical wire, and that, as the deviation of the wire from the vertical increases, both the actual distance s and the vertical distance

decrease. The distance for each bead can be measured, and from these measurements can be determined the effect on the motion which is caused by a given deviation from the vertical.

The distance which the bead on the vertical wire traverses during the first second is, as Galileo's experiment showed, equal to the distance through which any other freely falling body falls during the first second (when it falls from rest). This distance has been measured with great care many times and found to be approximately 16.1 ft. This is a number which occurs repeatedly in problems of motion, and for convenience is usually denoted by $\frac{1}{2}g$.* Its double, g, whose value is 32.2, is called the **acceleration of gravity,** for reasons which will appear later. Hence the distance traversed by the vertical bead for any value of t can be computed from the equation

$$s=\tfrac{1}{2}gt^2 ,$$

while for the beads on the slanting wires $s=dt^2$. The value of the symbol d is never as large as $\frac{1}{2}g$ and diminishes† as the deviation from the vertical increases.

The equations not only suffice for computing the distance when the time is known but may be used to compute the time when the distance is known. We then write, for the freely falling body,

$$\tfrac{1}{2}gt^2=s , \qquad gt^2=2s , \qquad t^2=2s/g , \qquad t=\sqrt{2s/g} .$$

For example, you could fall a mile in less than $\frac{1}{2}$ min.—in fact, in less than 26 sec. This seems surprising, since the fall is only 16.1 ft. the first second. Again, we can find the distance an object would fall the twentieth second by subtracting the total distance the first 19 sec., $\frac{1}{2}g(19)^2$ from the total distance the first 20 sec., $\frac{1}{2}g(20)^2$. This gives approximately 314 ft.

In a similar manner the mathematician is able to start with the Law of Inverse Squares and describe the motion of the earth around the sun, the motion of the moon around the earth, the tidal effect of the moon on

* Just as π is used for the ratio of the circumference of a circle to its diameter.

† A bead on a slanting wire is subject to two forces, viz., the force of gravity which pulls down and the push of the wire against the bead. The wire pushes in the direction perpendicular to it. The effective force which produces the motion is the resultant of the two forces (see chap. 4, sec. 7).

the earth, etc. He can show you how much higher you could jump if you were on the moon than you can jump on the earth, and can prove to you that if you were on one of the little moons of Mars you could jump high enough to keep on going.

Man jumps from small moon of Mars

The preceding discussion of the motion of an object which is acted upon by the force of gravity may seem to have led us far from the subject of this section, "The Abstract Nature of Mathematics"; but, indeed, we are still on the subject. The object of these remarks is to emphasize the fact already mentioned, viz., when the mathematician observes that a number of phenomena have a common property, he tries to find a symbolic expression for this common property which expresses the property itself apart from any of the specific objects to which it is attached. He then studies the formula as a purely mathematical product; he is interested in the relation of the symbols which appear in it. He asks himself many questions concerning the mathematical (not the physical) implications of the relationship. When he has a complete understanding of his algebraic relations, he is able to make the objective applications to any of the phenomena to which its origin was due; and moreover, he is quite likely to discover that he is now able to explain, in whole or in part, other phenomena not hitherto thought of as having anything in common with the first group considered.

A second possible result of such a study of a formula or a set of formulas as mathematical entities apart from any physical or geometrical significance must not be overlooked. Not infrequently the development of a purely mathematical theory leads on and on until mathematical situations are developed which have no apparent physical or geometrical explanation. It may or it may not happen that within a decade or a lifetime or a century this theory will be found to be of use in interpreting life about us. An instance of such a situation which is of considerable importance is the development of Einstein's theory of relativity. After plane and solid geometry had been developed, certain geometers went further and developed in detail the geometry in a space of four dimensions and even in a space whose dimension was designated by n, where n was supposed to be an arbitrary positive integer. Now this geometry, when developed analytically (i.e., by means of algebraic formulas), without regard to a visual or imaginative concept as to the makeup of a physical space of four or more dimensions, offers no particular difficulties; and it was this phase of it which interested the mathematician, though the non-mathematician was considerably agitated by the fact that he could not visualize a space of four dimensions and consequently he was certain that no such space exists and that there is something very peculiar about mathematicians who use their time developing theories about it.

Einstein's theory uses the mathematical set-up which is called "the geometry of a space of four dimensions," and it is quite reasonable to believe that the study of the physical concepts involved in this theory could not possibly have preceded the development of the mathematical theory. Einstein, as well as Euclid and Newton, found the way prepared before him.

2. Deductive reasoning. To reason is to draw inferences or conclusions from propositions (statements). The propositions on which the inferences are based may be the result of observations or they may be products of the mind. In the former case, the reasoning is generally called **inductive**; in the latter case, it is called **deductive.**

An excellent *example of inductive reasoning* is the already-mentioned Law of Gravitation of Newton. From his observations, whether in the open or in the laboratory, he concluded that there is always an attraction between two objects, that the attraction results in motion if the objects are free to move, that the attraction depends directly on the masses of the two objects and inversely on the square of their distance apart, and that motion takes place in the line joining the two centers of gravity. An induction always adds something to the observations. Newton's observations could not furnish him with such a precise statement as the inverse square law. That was his own contribution, his guess. How fortunate for him and for us that it turned out to be a satisfactory guess, because, of all hypothetical laws of attraction which have been studied by the mathe-