Lecture Notes in Mathematics

A collection of informal reports and seminars

Edited by A. Dold, Heidelberg and B. Eckmann, Zürich

233

Chris P. Tsokos

W. J. Padgett

Random Integral Equations with Applications to Stochastic Systems

Springer-Verlag

Berlin · Heidelberg · New York

Lecture Notes in Mathematics

A collection of informal reports and seminars
Edited by A. Dold, Heidelberg and B. Eckmann, Zürich

233

Chris P. Tsokos
Virginia Polytechnic Institute and State University,
Blacksburg, VA/USA

W. J. Padgett
University of South Carolina, Columbia, SC/USA

Random Integral Equations with Applications to Stochastic Systems

Springer-Verlag
Berlin · Heidelberg · New York 1971

AMS Subject Classifications (1970): 60 H 20, 93 E 99

ISBN 3-540-05660-2 Springer-Verlag Berlin · Heidelberg · New York
ISBN 0-387-05660-2 Springer-Verlag New York · Heidelberg · Berlin

Offsetdruck: Julius Beltz, Hemsbach / Bergstr.

PREFACE

Over the past few years we have been engaged in research concerning random or stochastic integral equations and their applications. A general theory of random integral equations of the Volterra and Fredholm types has been developed utilizing the theory of "admissibility" of spaces of functions and fixed-point methods of probabilistic functional analysis. We have two main objectives in these notes. First, we wish to give a complete presentation of the theory of existence and uniqueness of random solutions of the most general random Volterra and Fredholm equations which have been studied heretofore. The second objective is to emphasize the application of our theory to stochastic systems which have not been extensively studied before this time due to the mathematical difficulties that arise.

These notes will be of value to mathematicians, probabilists, and engineers who are working in the area of systems theory or to those who are merely interested in the theory of random equations.

It is anticipated that we will expand these notes to include other types of stochastic integral equations, such as the Hammerstein type and Ito's equation, along with many other applications in the general areas of engineering, biology, chemistry, and physics. We hope to reach this goal by 1972.

Chris P. Tsokos
Blacksburg, Va.
June, 1971

W. J. Padgett
Columbia, South Carolina
June, 1971

TABLE OF CONTENTS

GENERAL INTRODUCTION 1

I. PRELIMINARIES. 8

1.0 INTRODUCTION. 8

1.1 BASIC MATHEMATICAL CONCEPTS 8

1.2 THE STOCHASTIC INTEGRAL EQUATIONS 15

1.3 STOCHASTIC DIFFERENTIAL SYSTEMS 25

II. A RANDOM INTEGRAL EQUATION OF THE VOLTERRA TYPE. 27

2.0 INTRODUCTION. 27

2.1 SOME PRELIMINARY LEMMAS 28

2.2 EXISTENCE AND UNIQUENESS OF A RANDOM SOLUTION OF (2.0.1) 30

2.3 SOME SPECIAL CASES. 34

2.4 ASYMPTOTIC BEHAVIOR OF THE RANDOM SOLUTION. 41

2.5 APPLICATION TO THE POINCARÉ-LYAPUNOV STABILITY THEOREM 43

III. APPROXIMATE SOLUTIONS OF THE RANDOM VOLTERRA INTEGRAL EQUATION 48

3.0 INTRODUCTION. 48

3.1 THE METHOD OF SUCCESSIVE APPROXIMATIONS 49

3.1.1 Almost Sure Convergence of Successive Approximations 51

3.1.2 Rate of Convergence and Mean Square Error of Approximation. 56

3.1.3 Combined Error of Approximation and Numerical Integration. 59

3.2 THE METHOD OF STOCHASTIC APPROXIMATION. 66

3.2.1 A Stochastic Approximation Procedure 66

3.2.2 Solution of the Random Volterra Equation by Stochastic Approximation. 68

IV. A STOCHASTIC INTEGRAL EQUATION OF THE FREDHOLM TYPE WITH APPLICATION TO SYSTEMS THEORY. 76

4.0 INTRODUCTION. 76

4.1 EXISTENCE AND UNIQUENESS OF A RANDOM SOLUTION 77

4.2 SOME SPECIAL CASES. 92

4.3 STOCHASTIC ASYMPTOTIC STABILITY OF THE RANDOM SOLUTION. 96

4.4 AN APPLICATION IN STOCHASTIC CONTROL SYSTEMS. 99

V. RANDOM DISCRETE FREDHOLM AND VOLTERRA EQUATIONS. 106

5.0 INTRODUCTION. .106

5.1 EXISTENCE AND UNIQUENESS OF A RANDOM SOLUTION 107

5.2 SPECIAL CASES OF THEOREM 5.1.3. 112

5.3 STOCHASTIC STABILITY OF THE RANDOM SOLUTION 116

5.4 APPLICATION TO STOCHASTIC SYSTEMS 120

VI. THE STOCHASTIC DIFFERENTIAL SYSTEMS

$\dot{x}(t;\omega) = A(\omega)x(t;\omega) + b(\omega)\phi(\sigma(t;\omega))$

WITH

$\sigma(t;\omega) = \langle c(t;\omega), x(t;\omega)\rangle$

AND

$\dot{x}(t;\omega) = A(\omega)x(t;\omega) + b(\omega)\phi(\sigma(t;\omega))$

WITH

$\sigma(t;\omega) = f(t;\omega) + \int_0^t \langle c(t-\tau;\omega), x(\tau;\omega)\rangle d\tau$ 130

6.0 INTRODUCTION. 130

6.1 REDUCTION OF THE STOCHASTIC DIFFERENTIAL SYSTEMS. . .132

6.2 STOCHASTIC ABSOLUTE STABILITY OF THE SYSTEMS. 134

VII. THE STOCHASTIC DIFFERENTIAL SYSTEMS

$\dot{x}(t;\omega) = A(\omega)x(t;\omega) + \int_0^t b(t-\eta;\omega)\phi(\sigma(\eta;\omega))d\eta$

WITH

$\sigma(t;\omega) = f(t;\omega) + \int_0^t \langle c(t-\eta;\omega), x(\eta;\omega)\rangle d\eta$

AND

$\dot{x}(t;\omega) = A(\omega)x(t;\omega) + \int_0^t b(t-\tau;\omega)\phi(\sigma(\tau;\omega))d\tau + \int_0^t c(t-\tau;\omega)\sigma(\tau;\omega)d\tau$

WITH

$\sigma(t;\omega) = f(t;\omega) + \int_0^t \langle d(t-\tau;\omega), x(\tau;\omega)\rangle d\tau$ 144

7.0 INTRODUCTION. 144

7.1 REDUCTION OF THE STOCHASTIC DIFFERENTIAL SYSTEMS. . .145

7.2 STOCHASTIC ABSOLUTE STABILITY OF THE SYSTEMS. 149

VIII. THE STOCHASTIC DIFFERENTIAL SYSTEMS WITH LAG TIME

$\dot{x}(t;\omega) = A(\omega)x(t;\omega) + B(\omega)x(t-\tau;\omega) + b(\omega)\phi(\sigma(t;\omega))$

WITH

$\sigma(t;\omega) = f(t;\omega) + \int_0^t \langle c(t-s;\omega), x(s;\omega)\rangle ds$

AND

$\dot{x}(t;\omega) = A(\omega)\ x(t;\omega) + B(\omega)x(t-\tau;\omega) + \int_0^t \eta(t-u;\omega)\phi(\sigma(u;\omega))du + b(\omega)\phi(\sigma(t;\omega))$

WITH

$\sigma(t;\omega) = f(t;\omega) + \int_0^t \langle c(u;\omega), x(t-u;\omega)\rangle du$ 156

8.0 INTRODUCTION. 156

8.1 REDUCTION OF THE STOCHASTIC DIFFERENTIAL SYSTEMS. . .158

8.2 STOCHASTIC ABSOLUTE STABILITY OF THE SYSTEMS. 161

REFERENCES 166

GENERAL INTRODUCTION

The aim of these notes is to introduce the theory of random or stochastic integral equations of the Volterra and Fredholm types and to apply the results to certain general problems in systems theory. We hope to convey the manner in which such equations arise and to develop some general theory using as tools the methods of probability theory, functional analysis, and topology.

Due to the nondeterministic nature of phenomena in the general areas of the engineering, biological, oceanographic, and physical sciences, the mathematical descriptions of such phenomena frequently result in random or stochastic equations. These equations arise in various ways, and in order to understand better the importance of developing the theory of such equations and its application, it is of interest to consider how they arise.

Usually the mathematical models or equations that describe physical phenomena contain parameters or coefficients which have specific physical interpretations, but whose values are unknown. As examples, we have the volume-scattering coefficient in underwater acoustics, the coefficient of viscosity in fluid mechanics, the coefficient of diffusion in the theory of diffusion, and the modulus of elasticity in the theory of elasticity. Many times this unknown value is regarded as the true state of nature and is estimated by using the mean value of a set of observations obtained experimentally. The equations in the mathematical model are then solved in terms of the estimate of the unknown parameter or coefficient. If several sets of observations are obtained and the mean value is computed for each, then the mean values will most likely differ, and the particular mean value used as an estimate of the parameter may be quite unsatisfactory due to this random variation. Therefore, the parameter may be thought of as a random variable whose behavior is governed by some probability distribution function. Then, realistically, the equation must be viewed as a random equation, and its random solution must be obtained. Once such a solution is obtained its statistical properties should be studied.

There are many other ways in which random or stochastic equations arise. Stochastic differential equations appear in the study of diffusion processes and Brownian motion, Gikhmann and Skorokhod [1]. The classical Ito random integral equation (Ito [1]), which is a Stieltjes integral with respect to the Brownian motion process, may be found in many texts, for example, in Doob [1]. Integral equations with random kernels arise in random eigenvalue problems, Bharucha-Reid [6]. Stochastic integral equations describe wave propagation in random media, Bharucha-Reid [6], [7], and the total number of conversations held at a given time in telephone traffic theory, Fortet [1], Padgett and Tsokos [4]. In the theory of statistical turbulence, stochastic integral equations arise in describing the motion of a point in a continuous fluid in turbulent motion, Lumley [1], Padgett and Tsokos [3], Bharucha-Reid [7]. Integral equations were used by Bellman, Jacquez, and Kalaba [1], [2], [3] in a deterministic sense in the development of mathematical models for chemotherapy. However, due to the random nature of diffusion processes from the blood plasma into the body tissue, the stochastic versions of these equations are more realistic and should be used, Padgett and Tsokos [1], [2]. Stochastic or random equations also arise in systems theory, for example, Morozan [1], [2], [3], [4] and Tsokos [1], [2], [3], [4], [5].

Begun by A. Spacek in Czechoslovakia, there have been recent attempts by many scientists and mathematicians to develop and unify the theory of random equations utilizing the concepts and methods of probability theory and functional analysis, Adomian [1], Ahmed [1], Anderson [1], Bharucha-Reid [1], [2], [3], [4], [5], [6], [7], Bharucha-Reid and Arnold [1], Hans [1], Tsokos [4], Dawson [1]. Bharucha-Reid [5] refers to probabilistic functional analysis as being concerned with the application of the concepts and methods of functional analysis to the study of the various processes and structures which arise in the theory of probability and its applications.

Random or stochastic equations as described above may be categorized into four main classes as follows:

(i) Random or stochastic algebraic equations;

(ii) Random differential equations;

(iii) Random difference equations;

(iv) Random or stochastic integral equations.

In these notes we will be concerned with some classes of random or stochastic integral equations. In particular, we will be concerned with classes of stochastic integral equations of the Volterra type and of the Fredholm type. Specifically, we will investigate certain aspects of stochastic integral equations of the Volterra type of the form

$$x(t;\omega) = h(t;\omega) + \int_0^t k(t,\tau;\omega)f(\tau,x(\tau;\omega))d\tau \qquad (0.1)$$

and stochastic integral equations of the Fredholm type of the form

$$x(t;\omega) = h(t;\omega) + \int_0^\infty k_0(t,\tau;\omega)e(\tau,x(\tau;\omega))d\tau. \qquad (0.2)$$

We also will consider a discrete version of the stochastic integral equation (0.2) of the form

$$x_n(\omega) = h_n(\omega) + \sum_{j=1}^{\infty} c_{n,j}(\omega)\ e_j(x_j(\omega)).$$

The discrete version of equation (0.1) is then obtained as a special case of the above random discrete equation whenever

$$c_{n,j}(\omega) = \begin{cases} c^*_{n,j}(\omega), & j = 1,2,\ldots,n \\ 0 & , \text{otherwise.} \end{cases}$$

That is, the discrete version of the random integral equation (0.1) is

$$x_n(\omega) = h_n(\omega) + \sum_{j=1}^{n} c^*_{n,j}(\omega) f_j(x_j(\omega)) \quad .$$

In these notes we will be concerned primarily with the existence, uniqueness, and asymptotic behavior of random solutions of the equations (0.1) and (0.2) and their discrete analogs. We also will consider the approximation of the random solution of equation (0.1).

Equations (0.1) and (0.2) are more general than any random Volterra or Fredholm integral equations of these forms that have been considered to date. The generality consists primarily in the choice of the stochastic kernel and the nonlinearity of the equations. We also present the results for the random integral equations on a noncompact interval, whereas Anderson [1] was concerned only with equations whose functions were defined on compact intervals. These notes include the recent work of the authors, Padgett and Tsokos [5], [6], [7], [8], [9], Tsokos [3], [4], [5], and generalize the work of O. Hans [1], A. T. Bharucha-Reid [1], [2], [3], [4], and M. W. Anderson [1].

The second part of these notes is concerned with the application of the general results which are presented for the integral equation (0.1) to certain recently solved problems in stochastic differential systems, Morozan [1] and Tsokos [1], [2], [5]. These problems are as follows:

$$\dot{x}(t;\omega) = A(\omega)x(t;\omega) + b(\omega)\phi(\sigma(t;\omega))$$

with (0.3)

$$\sigma(t;\omega) = \langle c(t;\omega), x(t;\omega)\rangle \quad ;$$

$$\dot{x}(t;\omega) = A(\omega)x(t;\omega) + b(\omega)\phi(\sigma(t;\omega))$$

with (0.4)

$$\sigma(t;\omega) = f(t;\omega) + \int_0^t \langle c(t-\tau;\omega), x(\tau;\omega)\rangle d\tau;$$

$$\dot{x}(t;\omega) = A(\omega)x(t;\omega) + \int_0^t b(t-\tau;\omega)\phi(\sigma(\tau;\omega))d\tau$$

with (0.5)

$$\sigma(t;\omega) = f(t;\omega) + \int_0^t \langle c(t-\tau;\omega), x(\tau;\omega)\rangle d\tau;$$

$$\dot{x}(t;\omega) = A(\omega)x(t;\omega) + \int_0^t b(t-\tau;\omega)\phi(\sigma(\tau;\omega))d\tau + \int_0^t c(t-\tau;\omega)\sigma(\tau;\omega)d\tau$$

with (0.6)

$$\sigma(t;\omega) = f(t;\omega) + \int_0^t \langle d(t-\tau;\omega), x(\tau;\omega)\rangle d\tau;$$

$$\dot{x}(t;\omega) = A(\omega)x(t;\omega) + B(\omega)x(t-\tau;\omega) + b(\omega)\phi(\sigma(t;\omega))$$

with (0.7)

$$\sigma(t;\omega) = f(t;\omega) + \int_0^t \langle c(t-s;\omega), x(s;\omega)\rangle ds;$$

and

$$\dot{x}(t;\omega) = A(\omega)x(t;\omega) + B(\omega)x(t-\tau;\omega) + \int_0^t \eta(t-u;\omega)\phi(\sigma(u;\omega))du + b(\omega)\phi(\sigma(t;\omega))$$

with (0.8)

$$\sigma(t;\omega) = f(t;\omega) + \int_0^t \langle c(u;\omega),\ x(t-u;\omega)\rangle du.$$

These nonlinear stochastic differential systems will be reduced in a unified way to nonlinear stochastic integral equations of the Volterra type which are special cases of the random integral equation (0.1). In addition to the existence theory presented in these notes we will investigate the concept of stochastic absolute stability and give conditions which imply that the nonlinear stochastic systems (0.3) through (0.8) are stochastically absolutely stable. This type of stability has been studied by many scientists with respect to differential control systems in the nonstochastic case, but until recently had not been utilized for random systems.

The concept of absolute stability arose in the context of differential control systems and the general theory of stability of motion. The primary tool which was used until the late 1950's was Lyapunov's direct method. Then V. M. Popov developed a new approach which was called Popov's frequency response method. In these notes we successfully utilize the frequency response method with a random parameter to investigate the stochastic absolute stability of the stochastic differential systems (0.3), (0.4), (0.5), (0.6), and the systems (0.7) and (0.8) with lag time. System (0.3) was quite recently investigated by Morozan [1]. He reduced the system into a special case of the Volterra equation (0.1) and considered only a stochastic kernel in the exponential form. We will present a generalization of his results by considering the stochastic kernel in its most general form, Tsokos [5]. Stochastic differential systems of the remaining forms (0.4), (0.5), (0.6), (0.7), and (0.8) were open problems until solved recently by Tsokos [1], [2], [3], [4], [5], [6], [7], [8].

In Chapter I we shall present the preliminary mathematical and probabilistic definitions, notations, lemmas, and theorems which will be essential to the development of the theoretical results. Also, we shall formulate the stochastic integral equations (0.1) and (0.2). In Chapter II we will consider the random integral equation of the Volterra type with respect to the existence, uniqueness, and asymptotic properties of a random solution. Also, the results will be applied to a generalization of the Poincaré-Lyapunov stability theorem. Chapter III will be concerned with the theoretical approximation of the random solution of the Volterra equation (0.1). The random Fredholm integral equation (0.2) will be studied in Chapter IV with respect to the existence, uniqueness, and asymptotic properties of a random solution, and the results will be applied to a stochastic system. In Chapter V we will present the results concerning the random discrete Fredholm and Volterra systems, that is, the discrete versions of equations (0.1) and (0.2), and apply the results to some discrete stochastic systems. Chapters VI, VII, and VIII will be concerned with the stochastic absolute stability of the systems (0.3)-(0.8).

Some very recent results on the subject area have been obtained by C. P. Tsokos and A. N. V. Rao, [1], S. J. Milton and C. P. Tsokos, [1], [2], [3], [4] and S. J. Milton, W. J. Padgett and C. P. Tsokos, [1].

CHAPTER I

PRELIMINARIES

1.0 INTRODUCTION

In an attempt to make these notes somewhat self-contained, one of the purposes of this chapter is to present some of the basic definitions and theorems from functional analysis which will be often used. This will enable the reader to review quickly the material involved. The second aim of this chapter is to formulate the general stochastic integral equations of the Volterra and Fredholm types to be studied and to give the probabilistic definitions and spaces of functions which will be needed. Some of the probabilistic definitions and spaces of functions are being introduced here for the first time.

1.1 BASIC MATHEMATICAL CONCEPTS

We now state the following basic definitions and theorems.

Definition 1.1.1 A real-valued measurable function $f(x)$ defined on an interval $[a,b]$ is called a square-summable function if

$$\int_a^b |f(x)|^2 dx < \infty.$$

We will designate the class of all square-summable functions by L_2.

Definition 1.1.2 A real number associated with $f \varepsilon L_2$, denoted by $||f||$, is defined by

$$||f|| = \{\int_a^b |f(x)|^2 \, dx\}^{\frac{1}{2}}$$

and is called the norm of f.

Definition 1.1.3 A sequence $f_1, f_2, \ldots$ of functions in L_2 is said to converge to an element $f \varepsilon L_2$, called the limit of the sequence, if for every $\varepsilon > 0$, there exists a positive integer N such that $n > N$ implies that

$$||f_n - f|| < \varepsilon.$$

Definition 1.1.4 A nonempty set H is called a metric space if to any pair of elements x,y of H there corresponds a real number $\rho(x,y)$ with the following properties:

(i) $\rho(x,y) \geq 0$ and $\rho(x,y) = 0$ if and only if $x = y$;

(ii) $\rho(x,y) = \rho(y,x)$;

(iii) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ for any x, y, z ε H (triangle inequality).

The real number $\rho(x,y)$ is called the distance from x to y.

Definition 1.1.5 A sequence $\{x_n\}$ of elements of a metric space is a Cauchy sequence if for every $\varepsilon > 0$ there exists an $N > 0$ such that whenever $n > N$ and $m > N$, we have

$$\rho(x_n, x_m) < \varepsilon.$$

Definition 1.1.6 A metric space H is called complete if every Cauchy sequence in H converges to an element of H.

Definition 1.1.7 A nonempty set H is a linear space if

(i) to every pair of elements x and y of H there corresponds a third element z ε H such that $z = x + y$, called the sum of x and y;

(ii) to every x ε H and every scalar α, there corresponds an element $\alpha x \in H$, called the <u>product</u> of α and x;

(iii) the operations described in (i) and (ii) have the following properties for every x, y, z $\in$ H and scalars α and β:

(1) $x+y = y+x$ (commutative);

(2) $(x+y)+z = x+(y+z)$ (associative);

(3) $x+y = x+z$ implies $y = z$;

(4) $1x = x$;

(5) $\alpha(\beta x) = (\alpha\beta)x$;

(6) $(\alpha+\beta)x = \alpha x+\beta x$;

(7) $\alpha(x+y) = \alpha x+\alpha y$.

The concept of a <u>semi-norm</u> of an element of a linear space is sometimes used to introduce a topology in a linear space of infinite dimension. A complete discussion of semi-norms may be found in Yosida [1]. We will give the definition here.

<u>Definition 1.1.8</u> A real-valued function $\pi(x)$ defined on a linear space H is called a <u>semi-norm</u> if it satisfies the following conditions:

(i) $\pi(x) \geq 0$ for all $x \in H$;

(ii) $\pi(x+y) \leq \pi(x) + \pi(y)$ (subadditivity);

(iii) $\pi(\alpha x) = |\alpha|\pi(x)$, where α is any scalar.

<u>Definition 1.1.9</u> A linear space H is said to be <u>normed</u> if to each $x \in H$ there corresponds a real number $||x||$, called the <u>norm</u> of the element x, which has the following properties for each $y \in H$ and every scalar α:

(i) $||x|| \geq 0$, and $||x|| = 0$ if and only if x is the zero element of H;

(ii) $||\alpha x|| = |\alpha|\cdot||x||$;

(iii) $||x+y|| \leq ||x|| + ||y||$.