Functions of a Complex Variable, Operational Calculus, and Stability Theory

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ФУНКЦИИ КОМПЛЕКСНОГО ПЕРЕМЕННОГО
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ИЗДАТЕЛЬСТВО «НАУКА»

Functions of a Complex Variable, Operational Calculus, and Stability Theory

PROBLEMS AND EXERCISES

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Preface

The modern engineer often has to do with problems that require a sound mathematical background and set skills in the use of diverse mathematical methods. Expanding the mathematical outlook of the engineer contributes apprecia-bly to new advances in technology. The theory of functions of a complex variable, operational calculus, and stability theory are most important for applications.

The present text is designed for students of engineering colleges and also for practicing engineers who feel a need to refresh their knowledge of such important areas of mathe-matics. Each section of the text starts out with a brief review of the essentials of theory (propositions, definitions, formulas). This is followed by examples with detailed solutions and exercises. The book contains about 1000 exam-ples and problems to be solved by the student.

It would be impossible to acknowledge all the help, direct and indirect, that we have received in the preparation of this book, but certainly the many comments of Professor V.A. Trenogin, Head of the Mathematics Department at the Moscow Institute of Steel and Alloys, and of Associate Professor M.I. Orlov of the same department must be acknow-ledged. We are also greatful to members of the Applied Mathematics Department at the Kiev Engineering Con-struction Institute (head of department Associate Professor A.E. Zhuravel') and to B. Tkachev (Krasnodar) and

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Chapter 1

Functions of a Complex Variable

§ 1 Complex Numbers and Operations with Them

A complex number z is completely specified by a pair of real numbers x and y:

$$z = x + iy$$

(the *algebraic form* of a complex number), where $i^2 = -1$ (i is known as the *imaginary unit*). The number x is called the *real part* of the complex number and y the *imaginary part*. The notations are

$$x = \operatorname{Re} z, \quad y = \operatorname{Im} z.$$

The *conjugate* $\bar{z}$ of the complex number $z = x + iy$ is $\bar{z} = x - iy$. Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are assumed to be equal if and only if $x_1 = x_2$ and $y_1 = y_2$.

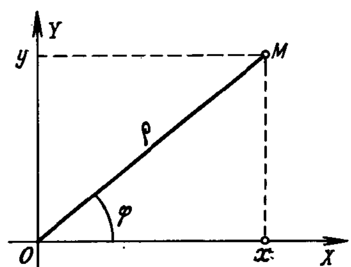

Figure 1

Any complex number $z = x + iy$ can be represented in a plane XOY (called the *complex (number) plane*, the *Gauss-Argand plane*, or the *Gaussian plane*) by either the point (x, y) (point M in Fig. 1) or the vector $\overline{OM}$ whose tail is at point O (0, 0) and tip at point M (x, y). The length ρ of vector $\overline{OM}$ is known as the *modulus* of the complex number and is denoted $|z|$, so that $\rho = |z| = (x^2 + y^2)^{1/2}$. The angle φ between the vector $\overline{OM}$ and the x axis is called the *argument* of z and is denoted $\varphi = \operatorname{Arg} z$; the argument of a complex number is determined only up to an integer multiple of 2π:

$$\operatorname{Arg} z = \arg z + 2k\pi \quad (k = 0, \pm 1, \pm 2, \ldots),$$

where $\arg z$ is the *principal value* of $\operatorname{Arg} z$, namely,

$$-\pi < \arg z < \pi,$$

with

$$\arg z = \begin{cases} \arctan(y/x) & \text{if} \quad x > 0, \\ \pi + \arctan(y/x) & \text{if} \quad x < 0 \quad \text{and} \quad y \geqslant 0, \\ -\pi + \arctan(y/x) & \text{if} \quad x < 0 \quad \text{and} \quad y < 0, \\ \pi/2 & \text{if} \quad x = 0 \quad \text{and} \quad y > 0, \\ -\pi/2 & \text{if} \quad x = 0 \quad \text{and} \quad y < 0. \end{cases} \qquad (1.1)$$

The following formulas are true:

$$\tan(\operatorname{Arg} z) = \frac{y}{x}, \quad \sin(\operatorname{Arg} z) = \frac{y}{\sqrt{x^2+y^2}},$$

$$\cos(\operatorname{Arg} z) = \frac{x}{\sqrt{x^2+y^2}}.$$

Two complex numbers z_1 and z_2 are considered equal if and only if their moduli are equal and their arguments are equal or differ by an ineger multiple of 2π:

$$|z_1| = |z_2|, \operatorname{Arg} z_1 = \operatorname{Arg} z_2 + 2\pi n \ (n = 0, \pm 1, \pm 2, \ldots).$$

Suppose we have two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

1. The *sum* $z_1 + z_2$ of z_1 and z_2 is a complex number defined thus:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2).$$

2. The *difference* $z_1 - z_2$ of z_1 and z_2 is defined as

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2).$$

3. The *product* $z_1 z_2$ of z_1 and z_2 is given by the following formula:

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

This definition yields, for one,

$$z\bar{z} = x^2 + y^2 = |z|^2.$$

4. The *quotient* z_1/z_2 of the division of z_1 by z_2 ($z_2 \neq 0$) is a complex number z such that $zz_2 = z_1$. For z_1/z_2 we have

$$\frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{|z_2|^2}. \qquad (1.2)$$

Here we used the formula $z_2^{-1} = \bar{z}_2/|z_2|^2$. We can also write (1.2) as

$$\frac{z_1}{z_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}.$$

The real part $\operatorname{Re} z$ and the imaginary part $\operatorname{Im} z$ of a complex number z can be expressed in terms of z and $\bar{z}$ thus:

$$\operatorname{Re} z = \frac{\bar{z} + z}{2}, \quad \operatorname{Im} z = i\,\frac{\bar{z} - z}{2} = \frac{z - \bar{z}}{2i}.$$

Example 1. Show that $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$.
Proof. By definition,

$$\overline{z_1 + z_2} = (x_1 + x_2) - i\,(y_1 + y_2) = (x_1 - iy_1) + (x_2 - iy_2) = \bar{z}_1 + \bar{z}_2.$$

1. Prove that

(a) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$; (b) $\overline{z_1z_2} = \bar{z}_1\bar{z}_2$; (c) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$;
(d) $\overline{\bar{z}_1 + \bar{z}_2} = z_1 + z_2$.

Example 2. Find the real solutions of the equation

$$(4 + 2i)\,x + (5 - 3i)\,y = 13 + i.$$

Solution. Separate the real and imaginary parts of the left-hand side of this equation, i.e. write $(4x + 5y) + i\,(2x - 3y) = 13 + i$. This, according to the definition of equality of two complex numbers, yields

$$\begin{cases} 4x + 5y = 13, \\ 2x - 3y = 1. \end{cases}$$

Solving this system of simultaneous equations, we find that

$$x = 2, \quad y = 1.$$

Find the real solutions of the following equations:

2. $(3x - i)(2 + i) + (x - iy)(1 + 2i) = 5 + 6i$.

3. $(x - iy)(a - ib) = i^5$, where a and b are given real numbers, and $|a| \neq |b|$.

4. $\frac{1}{z - i} + \frac{2 + i}{1 + i} = \sqrt{2}$, where $z = x + iy$.

5. Write the complex number $\frac{1}{(a+ib)^2}+\frac{1}{(a-ib)^2}$ in algebraic form.

6. Prove that $\frac{\sqrt{1+x^2}+ix}{x-i\sqrt{1+x^2}}=i$ (x is a real number).

7. Express x and y in terms of u and v if $\frac{1}{x+iy}+\frac{1}{u+iv}=1$ (x, y, u, and v are real numbers).

8. Find every complex number that satisfies the condition $\bar{z}=z^2$.

Example 3. Find the modulus and the argument of the complex number

$$z=-\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}.$$

Solution. We have

$$x=-\sin\frac{\pi}{8}<0,\quad y=-\cos\frac{\pi}{8}<0.$$

The principal value of the argument, according to (1.1), is

$$\arg z=-\pi+\arctan\left(\cot\frac{\pi}{8}\right)$$
$$=-\pi+\arctan\left[\tan\left(\frac{\pi}{2}-\frac{\pi}{8}\right)\right]$$
$$=-\pi+\arctan\left(\tan\frac{3\pi}{8}\right)=-\pi+\frac{3}{8}\pi=-\frac{5}{8}\pi.$$

Hence,

$$\operatorname{Arg} z=-\frac{5}{8}\pi+2k\pi\quad (k=0,\ \pm1,\ \pm2,\ \ldots),$$

$$|z|=\sqrt{\sin^2\frac{\pi}{8}+\cos^2\frac{\pi}{8}}=1.$$

9. In the problems below find the modulus and the principal value of the argument of the following complex numbers:

(a) $z=4+3i$; (b) $z=-2+2\sqrt{3}i$;

(c) $z=-7-i$; (d) $z=-\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}$;

(e) $z=4-3i$; (f) $z=\cos\alpha-i\sin\alpha$ $\left(\pi<\alpha<\frac{3}{2}\pi\right)$.

Each complex number $z = x + iy$ $(z \neq 0)$ can be written in *polar form*:

$$z = \rho(\cos\varphi + i\sin\varphi), \quad \text{where} \quad \rho = |z|, \quad \varphi = \operatorname{Arg} z.$$

Example 4. Write the complex number

$$z = -1 - i\sqrt{3}$$

in polar form.

Solution. We have

$$|z| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = 2; \quad \tan\varphi = \frac{-\sqrt{3}}{-1} = \sqrt{3},$$

$$\varphi = -\frac{2}{3}\pi.$$

Hence,

$$-1 - i\sqrt{3} = 2\left[\cos\left(-\frac{2}{3}\pi\right) + i\sin\left(-\frac{2}{3}\pi\right)\right].$$

Example 5. Find the real roots of the equation

$$\cos x + i\sin x = \frac{1}{2} + \frac{3}{4}i.$$

Solution. This equation has no roots. Indeed, the equation is equivalent to two equations: $\cos x = 1/2$ and $\sin x = 3/4$. But these equations cannot be satisfied simultaneously, since $\cos^2 x + \sin^2 x = 13/16$, which is impossible for any x.

Any complex number $z \neq 0$ can be written in exponential form:

$$z = \rho e^{i\varphi}, \quad \text{where} \quad \rho = |z|, \quad \varphi = \operatorname{Arg} z.$$

Example 6. Find all complex numbers $z \neq 0$ that satisfy the condition $z^{n-1} = \bar{z}$.

Solution. Let $z = \rho e^{i\varphi}$. Then $\bar{z} = \rho e^{-i\varphi}$. Since according to the hypothesis

$$\rho^{n-1} e^{i(n-1)\varphi} = \rho e^{-i\varphi}, \quad \text{or} \quad \rho^{n-2} e^{in\varphi} = 1,$$

we find that $\rho^{n-2} = 1$, i.e. $\rho = 1$, and $in\varphi = 2k\pi i$, i.e. $\varphi = 2k\pi/n$ $(k = 0, 1, \ldots, n-1)$. Hence,

$$z_k = e^{i(2\pi k/n)} \quad (k = 0, 1, 2, \ldots, n-1).$$

10. Represent the following complex numbers in polar form:

(a) -2; (b) $2i$; (c) $-\sqrt{2} + i\sqrt{2}$;

(d) $1 - \sin\alpha + i\cos\alpha \; \left(0 < \alpha < \frac{\pi}{2}\right)$;

(e) $\dfrac{1 + \cos\alpha + i\sin\alpha}{1 + \cos\alpha - i\sin\alpha} \; \left(0 < \alpha < \frac{\pi}{2}\right)$;

and in exponential form:

(f) -2; (g) i; (h) $-i$; (i) $-1 - i\sqrt{3}$;

(j) $\sin\alpha - i\cos\alpha$ $(\pi/2 < \alpha < \pi)$; (k) $5 + 3i$.

Suppose two complex numbers, z_1 and z_2, are given in polar form: $z_1 = \rho_1 (\cos\varphi_1 + i\sin\varphi_1)$ and $z_2 = \rho_2 (\cos\varphi_2 + i\sin\varphi_2)$.

Their product can be determined from the following formula:

$$z_1 z_2 = \rho_1\rho_2 [\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)],$$

i.e. if complex numbers are multiplied, their moduli are multiplied and their arguments added:

$$|z_1 z_2| = |z_1|\,|z_2|,$$
$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2.$$

The quotient of the division of complex number z_1 by complex number $z_2 \neq 0$ can be determined from the following formula:

$$\frac{z_1}{z_2} = \frac{\rho_1}{\rho_2}[\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)],$$

i.e.

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \quad \operatorname{Arg}\frac{z_1}{z_2} = \operatorname{Arg} z_1 - \operatorname{Arg} z_2.$$

Raising a complex number

$$z = \rho(\cos\varphi + i\sin\varphi)$$

to a positive integer n is done according to the formula

$$z^n = \rho^n (\cos n\varphi + i\sin n\varphi),$$

i.e.

$|z^n| = |z|^n$, $\operatorname{Arg} z^n = n \operatorname{Arg} z + 2\pi k$ $(k = 0, \pm 1, \ldots)$.

This gives *de Moivre's formula*

$$(\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi.$$

Properties of moduli of complex numbers

1. $|\bar z| = |z|$; 2. $z\bar z = |z|^2$; 3. $|z_1 z_2| = |z_1|\,|z_2|$;

4. $|z^n| = |z|^n$; 5. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \neq 0$;

6. $|\operatorname{Re} z| \leqslant |z|$, $|\operatorname{Im} z| \leqslant |z|$; 7. $|z_1 + z_2| \leqslant |z_1| + |z_2|$; 8. $||z_1| - |z_2|| \leqslant |z_1 - z_2|$.

Example 7. Evaluate $(-1 + i\sqrt{3})^{60}$.

Solution. Represent $z = -1 + i\sqrt{3}$ in polar form:

$$-1 + i\sqrt{3} = 2\left(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi\right).$$

Applying the above formula for raising to a power yields

$$(-1 + i\sqrt{3})^{60} = 2^{60}\left[\cos\left(60\cdot\frac{2}{3}\pi\right) + i\sin\left(60\cdot\frac{2}{3}\pi\right)\right]$$
$$= 2^{60}(\cos 40\pi + i\sin 40\pi) = 2^{60}.$$

Example 8. Prove that the polynomial

$$f(x) = (\cos\alpha + x\sin\alpha)^n - \cos n\alpha - x\sin n\alpha$$

is divisible by $x^2 + 1$.

Solution. Since $x^2 + 1 = (x + i)(x - i)$, de Moivre's formula yields

$$f(i) = (\cos\alpha + i\sin\alpha)^n - \cos n\alpha - i\sin n\alpha$$
$$= \cos n\alpha + i\sin n\alpha - \cos n\alpha - i\sin n\alpha = 0.$$

Similarly, $f(-i) = 0$. This means that $f(x)$ is divisible by $x^2 + 1$.

11. Prove that the polynomial

$$f(x) = x^n\sin\alpha - \lambda^{n-1}x\sin n\alpha + \lambda^n\sin(n-1)\alpha$$

is divisible by $x^2 - 2\lambda x\cos\alpha + \lambda^2$.

12. Evaluate:

(a) $\left(\frac{1+i\sqrt{3}}{1-i}\right)^{40}$; (b) $(2-2i)^7$;

(d) $\left(\frac{1-i}{1+i}\right)^8$.

13. Prove that

$$\left(\frac{1+i\tan\alpha)}{1-i\tan\alpha}\right)^n = \frac{1+i\tan n\alpha}{1}$$

14. Prove that if

$(\cos\alpha + i\sin\alpha)^n = 1$, then

15. Using de Moivre's formula, express in terms of powers of $\sin\varphi$ and $\cos\varphi$ the following functions of integer multiples of φ:

(a) $\sin 3\varphi$; (b) $\cos 3\varphi$; (c) $\sin 4\varphi$; (d) $\cos 4\varphi$; (e) $\sin 5\varphi$; and (f) $\cos 5\varphi$.

An nth root of a complex number z has n different values given by the formula

$$\sqrt[n]{z}=\sqrt[n]{|z|}\left(\cos\frac{\varphi+2k\pi}{n}+i\sin\frac{\varphi+2k\pi}{n}\right),$$

where $k=0, 1, 2, \ldots, n-1$, and $\varphi=\arg z$. The points corresponding to the values of $\sqrt[n]{z}$ are vertices of the regular n-gon inscribed in the circle of radius $R=\sqrt[n]{|z|}$ centered at the origin of coordinates.

The nth root of a real number a also has n different values; among these there are two real roots, one, or no real root, depending on whether n is even or odd and the sign of a.

Example 9. Find all values of $\sqrt[4]{1-i}$.

Solution. Reduce the complex number $1-i$ to polar form:

$$1-i=\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right].$$

Hence,

$$\sqrt[4]{1-i}=\sqrt[8]{2}\left(\cos\frac{-\frac{\pi}{4}+2k\pi}{4}+i\sin\frac{-\frac{\pi}{4}+2k\pi}{4}\right).$$

Putting $k=0, 1, 2, 3$ yields

$$(k=0)\quad \sqrt[4]{1-i}=\sqrt[8]{2}\left(\cos\frac{\pi}{16}-i\sin\frac{\pi}{16}\right),$$

$$(k=1)\quad \sqrt[4]{1-i}=\sqrt[8]{2}\left(\cos\frac{7}{16}\pi+i\sin\frac{7}{16}\pi\right),$$

$$(k=2)\quad \sqrt[4]{1-i}=\sqrt[8]{2}\left(\cos\frac{15}{16}\pi+i\sin\frac{15}{16}\pi\right),$$

$$(k=3)\quad \sqrt[4]{1-i}=\sqrt[8]{2}\left(\cos\frac{23}{16}\pi+i\sin\frac{23}{16}\pi\right).$$

In the following exercises find all values of the root:

16. (a) $\sqrt[4]{-1}$; (b) $\sqrt{i}$; (c) $\sqrt[3]{i}$; (d) $\sqrt[4]{-i}$.

17. (a) $\sqrt[4]{1}$; (b) $\sqrt[3]{-1+i}$; (c) $\sqrt{2-2\sqrt{3}i}$.

18. $\sqrt[5]{\sqrt{2}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)}$.

Example 10. What set of points in the complex z plane is defined by the condition

$$\operatorname{Im} z^2>2?$$