A FIRST COURSE IN NUMERICAL ANALYSIS

SECOND EDITION

ANTHONY RALSTON
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PREFACE

The 12 years since the publication of the first edition have been a time of great progress in numerical analysis. This progress is epitomized by the development in recent years of the first complete subroutine packages for digital computers—so-called *mathematical software*—programs which can accept as input the parameters of a problem in a particular area of numerical analysis and which will generally produce as output the solution to the problem within the accuracy desired (or, rarely, a statement that the problem is insoluble or not solvable to the desired accuracy) without the necessity for the user to choose the method of solution. This has been made possible by the development of methods—or classes of methods—which are responsive to all or nearly all the difficulties or sensitivities which can occur in numerical analytic problems.

It is hardly surprising, therefore, that the first edition is now badly out of date in many respects. In this edition we have brought all chapters in the book up to date as of 1977 and have deleted material no longer of interest because it has been superseded by more modern techniques. Some additional material has been deleted solely because of space limitations.

The scope of the book itself may be easily gleaned from the Table of Contents. Here we note only the major additions to and changes from the first edition.

Chapter 1. New sections on Norms, Error Analysis, and Condition and Stability; rewritten section on Computer Arithmetic.

Chapter 2. New sections on Numerical Algorithms, Functionals, and the Method of Undetermined Coefficients.

Chapter 3. New section on Splines.

Chapter 4. New sections on Adaptive Integration and the Euler Transformation; rewritten sections on Numerical Differentiation.

Chapter 5. New sections on Variable Order, Variable Step Methods, Extrapolation Methods, and Stiff Equations; revised section on Runge-Kutta Methods.

Chapter 6. New section on the Fast Fourier Transform.

Chapter 7. New section on the Differential Correction Algorithm.

Chapter 8. New sections on the Jenkins-Traub Method and a Newton-based Method for the zeros of polynomials; rewritten sections on Systems of Nonlinear Equations and on the general problem of the Zeros of Polynomials.

Chapter 9. New sections on Overdetermined Systems and the Simplex Method; rewritten sections on Direct Methods and Error Analysis.

Chapter 10. New sections on the Inverse Power Method and Jacobi-type Methods for nonsymmetric matrices; rewritten section on the *QR* Algorithm.

In addition, there are many changes in other sections to improve clarity and to reflect advances since the publication of the first edition.

It will ordinarily not be possible to cover all the material in this book in a full-year course in numerical analysis. Rather than suggest topics for inclusion or exclusion, we would leave this to the instructor's own taste and experience. The fact that the subjects of each of Chapters 3 through 10 have themselves been the subjects of at least one book apiece should serve to emphasize to the student that a course taught from this book is indeed a first course in numerical analysis. Moreover, we have not covered such topics as the numerical solution of partial differential equations, integral equations, or boundary-value problems. These topics properly fall in the domain of advanced numerical analysis. Since the basis of much of advanced numerical analysis is the solution of systems of linear equations and the calculation of eigenvalues, these topics have been purposely placed at the end of this volume.

In each of Chapters 3 through 10 there are a number of illustrative examples whose purpose is to enhance the student's understanding of the relevant numerical method. Since a morass of numbers is more likely to impede this aim than otherwise, the numbers in these examples have, where possible, been kept simple.

In this edition we have added problems at the end of each chapter corresponding to the new and rewritten sections. The problems fall generally into four categories:

1. Simple proofs of topics considered in the text.
2. Algebraic manipulations and derivations, which would not add materially to the understanding of the student if included in the text, but which may nevertheless be instructive.
3. Computational problems.

4. Proofs and derivations of results which are an extension of the subject matter in the text.

Although the especially difficult problems have been starred (*), the student will find few really easy problems. One of the major purposes of the separately published *Hints and Answers* is to help the student solve those problems found particularly difficult. The student should be prepared to find minor discrepancies between calculated numerical answers and those in this manual. These will generally be the result of the idiosyncrasies of roundoff error on the computer on which the calculations have been performed.

The few bibliographic references in the text itself are to topics outside the scope of numerical analysis or to topics not suitable for problems. The Bibliographic Notes and Bibliographies at the end of each chapter have been brought up to date for this edition. They are meant to guide the student to basic sources from which a deeper understanding of the subject matter of this book may be obtained. For this reason no attempt has been made to make the bibliography exhaustive, and comparatively few foreign-language references have been included.

Anthony Ralston
Philip Rabinowitz

NOTATION

Below is a list of symbols and notation used in this book. Amplified explanations are given when necessary at the first use of the symbol or notation. The reader is cautioned that some symbols may have more than one meaning but it is hoped that they are unambiguous; for example, $P_n(x)$ is used as the notation for the Legendre polynomial of degree n in Chap. 4 and thereafter, but in Chap. 2 this notation is also used for a general polynomial of degree n.

A. Problems and references

Meaning	Symbol or example	Page first defined or used
1. References in text to problems at end of each chapter: numbers in braces	{2}	5
2. References to bibliography at end of each chapter: name followed by date in parentheses	Feller (1950)	10
3. Problems of more than ordinary difficulty: asterisk next to problem number	***10**	26

B. General mathematical notation

1. Approximately equal	$\approx$	6
2. Binomial coefficient	$\binom{n}{k}$	36
	$(m)_k$	59

Meaning	Symbol or example	Page first defined or used
3. Closed interval	$[a,b]$	1
4. Conjugate transpose of vector or matrix: superscript asterisk*	$\mathbf{v}^*$	445
5. Continued fraction	$\frac{C_1\|}{\|x+D_1} + \frac{C_2\|}{\|x+D_2} + \cdots$	292
6. Derivatives:		
(a) Single, double, or triple prime	$f''(\xi_1)$	1
(b) Lowercase roman superscript	$f^{\mathrm{iv}}(\xi_2)$	1
(c) Letter or number in parentheses	$f^{(n)}(x)$	43
7. Determinant of a matrix	$\|A\|$	412
	$\det(L_{n-1})$	423
8. Evaluation of a quantity at a point	$\|_n$	210
9. Factorial function	$x^{(n)}$	141
10. Functional	$F(f)$	42
11. Inner product		
(a) T for transpose	$\mathbf{v}^T\mathbf{v}$	6
(b) * for conjugate transpose	$\mathbf{x}^*\mathbf{x}$	487
12. Integer functions		
(a) Ceiling	$\lceil x \rceil$	15
(b) Floor	$\lfloor x \rfloor$	15
13. Norm		
(a) function: L_p	$\\|f\\|_p$	7
(b) matrix: general	$\\|A\\|$	7
(c) matrix: Euclidean	$\\|A\\|_E$	8
(d) vector: L_p	$\\|\mathbf{v}\\|_p$	7
(e) vector: Euclidean	$\\|\mathbf{v}\\|$	6
14. Open interval	(a_i, m_i)	40
15. Order of magnitude	$O(h^2)$	174
16. Sequences of functions or numbers: indexed quantity in braces (cf. A1 above)	$\{x^n\}$	33
17. Spectral radius of a matrix	$\rho(A)$	8
18. Transform pair	$G_j \leftrightarrow g_k$	264
19. Vector: boldface English or Greek letters		
(a) Column	$\mathbf{v}$	6
(b) Row (T for transpose)	$\mathbf{v}^T$	6
20. Vector function	$\mathbf{f} = [f_1 f_2 \cdots f_n]$	359

C. Specific mathematical symbols

Symbol	Meaning	Page first defined or used
1. $B_n(x)$	Bernstein polynomial	3
2. $B_n(x)$	Bernoulli polynomial	136
3. B_n	Bernoulli number	136

Symbol	Meaning	Page first defined or used
4. EI	Efficiency index	337
5. erf(x)	Error function	26
6. $H_n(x)$	Hermite polynomial	107
7. I	Identity matrix	433
8. $I_a^b R(x)$	Cauchy index	368
9. $J_n(x; \alpha, \beta)$	Jacobi polynomial	108
10. $l_j(x)$	Lagrangian interpolation polynomial	53
11. $L_n(x)$	Laguerre polynomial	106
12. $P_n(x)$	Legendre polynomial	99
13. $S_n(x)$	Chebyshev polynomial of second kind	111
14. T	Transpose of matrix or column vector	6
15. $T_n(x)$	Chebyshev polynomial	109
16. $T_r^*(x)$	Shifted Chebyshev polynomial	327
17. tr (A)	Trace of a matrix	26
18. $\mathbf{x}_c$	Computed solution of linear system	413
19. $\mathbf{x}_t$	True solution of linear system	413
20. ∇	Backward-difference operator	58
21. ∇	Gradient operator	362
22. Δ	Forward-difference operator	58
23. δ	Central-difference operator	58
24. δ_{jk}	Kronecker delta	54
25. $\Gamma(x)$	Gamma function	108
26. μ	Mean central-difference operator	83
27. $\cup$	Set union	486

CONTENTS

CHAPTER
ONE

INTRODUCTION AND PRELIMINARIES

1.1 WHAT IS NUMERICAL ANALYSIS?

That numerical analysis is both a science and an art is a cliché to specialists in the field but is often misunderstood by nonspecialists. Is calling it an art and a science only a euphemism to hide the fact that numerical analysis is not a sufficiently precise discipline to merit being called a science? Is it true that "numerical analysis" is something of a misnomer because the classical meaning of analysis in mathematics is not applicable to numerical work? In fact, the answer to both these questions is no. The juxtaposition of science and art is due instead to an uncertainty principle which often occurs in solving problems, namely that to determine the best way to solve a problem may require the solution of the problem itself. In other cases, the best way to solve a problem may depend upon a knowledge of the properties of the functions involved which is unobtainable either theoretically or practically. A simple example will illustrate this. Two common methods for estimating

$$\int_a^b f(x)\,dx$$

are the trapezoidal rule and the parabolic rule. The error incurred, i.e., the difference between the true value of the integral and the approximation, in the former is $-(b-a)^3 f''(\xi_1)/12n^2$, where $n+1$ is the number of points at which we evaluate $f(x)$ in $[a, b]$ and ξ_1 is some (unknown) point in $[a, b]$. For the parabolic rule the error is $-(b-a)^5 f^{\text{iv}}(\xi_2)/180n^4$, where again $n+1$ is the number of points at which $f(x)$ is evaluated and ξ_2 is an unknown