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Group Actions on Manifolds

Reinhard Schultz, Editor

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Group Actions on Manifolds

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HELD AT THE UNIVERSITY OF COLORADO, BOULDER
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INTRODUCTION

The 1983 A. M. S. Summer Research Conference series included a one week Conference on Group Actions on Manifolds from Monday, June 27, through Friday, July 1, 1983. This conference provided a setting for discussing recent work, and it also presented an opportunity to summarize the current state of the subject and review the progress made during the past two decades. There were general lectures during the mornings and in some cases also during the early afternoons. The rest of the afternoons and one evening were devoted to talks on more specialized topics. In addition there was a problem session and a tribute to Professor Deane Montgomery in recognition of his achievements and influence on the subject.

These conference proceedings contain written versions of many talks at the conference and some articles by other workers in the area who were not able to attend the conference. The articles are organized by subject matter rather than in alphabetical or chronological order. Expository articles generally appear before more specialized articles. The opening discussion of the work of D. Montgomery and the problem collection at the end have been extensively reorganized and supplemented.

The Organizing Committee for the conference consisted of Karl Heinz Dovermann, Ted Petrie, Frank Raymond, and myself. Each of my colleagues on the committee provided valuable advice and assistance with planning and organizational details, and of course I am grateful to them for their help. I would also like to thank all participants, referees, and supporting staff who contributed to the operations and activities of the conference and the preparation of these proceedings. Special thanks are due to A.M.S. coordinator Carole Kohanski for her continued patience and energy and to Karen Ruley at Purdue for efficiently handling large amounts of correspondence in connection with both the conference itself and these proceedings. Several other technical typists at Purdue (Sandy Emery, Kathy Johnson, Jackie Oswalt, Judy Snider, Elizabeth Young) also deserve thanks for their roles in preparing many of the manuscripts and some of the correspondence. Finally, the Department of Mathematics at Purdue University has been generous and patient in absorbing the costs of fairly large bills for xeroxing, postage, and long distance telephone calls; and I am of course grateful for this.

All papers in these proceedings were refereed and are in their final forms.

Reinhard Schultz
Purdue University
August, 1984

SUMMARY OF PROGRAM

Joint Summer Research Conference in the Mathematical Sciences

Group Actions on Manifolds

University of Colorado, Boulder, June 26 - July 1, 1983

The following is a list of talks presented at the conference. They are arranged by topic rather than chronologically. Abstracts for most of these talks appeared in the November, 1983, issue of *Abstracts of papers presented to the American Mathematical Society* (Volume 4, Number 7). The page numbers for the abstracts appear in parentheses after the titles of the talks.

Homotopy-theoretic techniques and applications

R. Schultz, "Homotopy invariants and G-manifolds: A look at the past 15 years" (p. 539).

S. Illman, "Whitehead torsion and actions of compact Lie groups" (p. 547).

G. Triantafillou, "Algebraic models for G-simple homotopy type" (p. 547).

R. Dotzel, "Solid torus splittings of semifree actions on spheres" (p. 542).

J. Oprea, "Lifting homotopy actions in rational homotopy theory" (p. 538).

G. Katz, "Normal discrete portraits of G-actions" (p. 541).

Homological methods and machinery

L. G. Lewis, "RO(G)-graded equivariant ordinary cohomology of a point" (p. 539).

M. Özaydın, "The trace invariant of a group action" (p. 540).

A. Necochea, "Borsuk-Ulam theorems for prime periodic transformation groups" (p. 544).

B. Mann, "Characteristic classes for equivariant spherical fibrations" (p. 539).

D. C. Royster, "Dimension of the fixed point set for semifree circle actions" (p. 545).

Applications of surgery theory and geometric topology

S. Weinberger, "Homologically trivial free actions of finite groups" (p. 540).

K. H. Dovermann, "Transformation groups and fixed point data" (p. 539).

M. Rothenberg, "Differences between some natural categories of G-manifolds."

R. Schultz, "Transformation groups and exotic spheres" (p. 540).

A. Assadi, "Realization questions for semifree actions of finite groups" (p. 542).

C. Stark, "K-theory and surgery of codimension-two torus actions on aspherical manifolds" (p. 540).

R. Lashof, "Reduction of the G-smoothing problem."

M. Rothenberg, "Compact locally smoothable G-manifolds with nonfinite G-homotopy type" (p. 546).

J. Davis, "The surgery semicharacteristics and applications" (p. 541).

F. Connolly, "An approach to equivariant surgery" (p. 548).

B. Levy, "A splitting theorem for some of M. Davis' manifolds" (p. 548).

R. Ball, "(n+1)-axial O(n), U(n), Sp(n) actions on homotopy spheres" (p. 543).

D. Y. Suh, "s-Smith equivalence for finite abelian groups" (p. 548).

E. C. Cho, "s-Smith equivalent representations of generalized quaternion groups" (p. 546).

Y. D. Tsai, "Isotropy representations of nonabelian groups on disks" (p. 545).

Low-dimensional topology and transformation groups

F. Raymond and A. Edmonds, "Transformation groups and low-dimensional topology [2 lectures]" (p. 542).

A. Edmonds, "Periodic classical knots" (p. 542).

Homogeneous spaces and Seifert fiberings

F. Raymond, "Transformation groups and Seifert fiberings" (p. 540).

K. B. Lee, "Applications of the Seifert fiber space construction" (p. 546).

R. Lee, "Finite group actions on locally symmetric spaces" (p. 544).

S. Weintraub, "Cohomology of Siegel modular varieties of degree two" (pp. 544 [Pt. I], 538 [Pt. II]).

H. Kim, "Complete left-invariant flat affine structures on nilpotent Lie groups" (p. 547).

Y. Kamishima, "Closed aspherical manifolds dominated by Lorentz forms and polycyclic groups" (p. 543).

Transformation groups and differential geometry

D. Burghelea, "Symmetry of manifolds, homotopy groups, and geometric structure" (p. 543).

H. T. Laquer, "Geometry, representation theory, and the Yang-Mills functional" (p. 538).

H. T. Ku, "The Hilbert-Smith Conjecture and Newman's Theorem" (p. 540).

Problem session

(An expanded account appears in these proceedings.)

LIST OF PARTICIPANTS

D. R. Anderson (Syracuse)
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CONTENTS

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THE WORK AND INFLUENCE OF DEANE MONTGOMERY

When Deane Montgomery began his mathematical career in the early nineteen thirties, the importance of symmetry in mathematics had already been recognized for at least a century. However, in those days the established techniques for studying symmetry were either algebraic or analytic in nature. This is not surprising, for much of the motivation for studying transformation groups originated in the linear representation theory of finite groups, the analytic mapping theory of compact Riemann surfaces, and Riemann's differential-geometric approach to the foundations of geometry. Furthermore, the work of Lie on analytic transformation groups had furnished a powerful method for studying many questions from differential geometry, complex analysis, and differential equations.

On the other hand, there was rapidly increasing evidence to indicate that topological techniques could also be powerful tools in studying symmetry questions. The possibility of developing a topological theory of transformation groups was discussed explicitly in the Fifth Problem of Hilbert's famous list [Hi 1] (some of Hilbert's results are also discussed below). Results of Brouwer, Kérékjárto, and others on periodic maps of surfaces had shown that some basic symmetry phenomena were essentially topological. Furthermore, topological methods had led to an improved understanding of Lie's concept of a continuous group; in particular, topology provided the right setting for passing from local groups to global groups. Conversely, it had become apparent that a great many locally compact topological groups could be viewed as limits of Lie groups. The researchers that made important contributions in these connections included Brouwer, Kérékjárto, van Kampen, E. Cartan, Freudenthal, von Neumann, and Pontrjagin. Finally, there were Newman's theorems on the diameters of orbits of a group action and the nowhere denseness of fixed point sets.

Despite these advances, one could not say that topological transformation groups was a well recognized discipline. Knowledge of the global structure of topological groups and much of topology were in very tentative states. Much of the terminology that we use today and many of the ideas we may casually toss off as routine were still not invented or formulated.

One of the rewards of looking through Montgomery's papers is the chance to view the creation and development of a subject in mathematics that has attracted the efforts of many excellent mathematicians. The papers lead the reader through the mainstream of activity over several decades and present ideas and techniques that have dominated and motivated research for years.

Montgomery's earliest papers on transformation groups may be divided into two parts. The first part deals with questions of a fairly general nature. For example, in [6] Montgomery shows that for a large class of topological groups the standard continuity axioms for the multiplication and inverse maps can be weakened substantially. In [8] Montgomery proves that a pointwise periodic self-map of a manifold is in fact periodic. This is somewhat surprising at first glance, for it is fairly easy to construct pointwise periodic self-homeomorphisms of nonmanifolds (in fact, for compact subsets of $\mathbb{R}^3$ with one singular point) that do not have finite order. The proof is an elegant application of Newman's earlier work.

The second part of Montgomery's early work on group actions is a sequence of papers with L. Zippin on the linearizability of group actions on low-dimensional manifolds ([5], [7], [11-13], [15]). One can view these results as a natural continuation of the work of Brouwer, Kérékjárto, and Eilenberg on the linearizability of group actions on curves and surfaces. However, it is also interesting and worthwhile to recognize the connections between these papers and the foundations of geometry. The possibility of characterizing classical geometry by topological transformation groups was mentioned fairly explicitly in Hilbert's own formulation of the fifth problem in [Hi 1]. In fact, shortly afterwards Hilbert described a characterization of this sort for Euclidean plane geometry [Hi 2]. This parallels an earlier characterization by S. Lie in terms of analytic transformation groups [Lie]. Since Hilbert's motivation for doing this has had a tremendous impact upon transformation groups, we shall quote from the first page of Hilbert's article:

> In developing his theory of transformation groups, Lie always assumed that the functions defining the groups can be differentiated, and thus in Lie's development there is no discussion of the question whether the assumption of differentiability as far as the geometric axioms are concerned

> is really indispensable, or whether the differentiability of the functions dealt with is rather a direct consequence of the group concept and the other geometric concepts. Because of his method Lie was obliged to formulate explicitly the axiom that the group of motions is generated by infinitesimal transformations. These requirements, as well as other essential parts of the axioms assumed by Lie with respect to the nature of the equation which defines equidistant points, can be expressed purely geometrically only by brute force and in a complicated way. Besides, they appear only through the analytic method used by Lie and are not due to the problem itself.
>
> Therefore, I have attempted ... to formulate a set of axioms for plane geometry which while resting on the concept of a group contain only simple and geometrically clear requirements, and in particular assume in no way the differentiability of the functions induced by the motions.

The results of [15] furnish a corresponding characterization of three-dimensional geometry by topological transformation groups. Related results were obtained by Kérékjárto; a survey of work on this topic is given in [Zip]. In [5] and [11] Montgomery and Zippin show that compact, connected, abelian groups which act on $\mathbb{R}^3$ are toral groups and the actions are equivalent to linear actions. This requires a proof that the group must be a Lie group and analysis of the actions of toral groups. The first part of this study amounts to a beginning of Montgomery and Zippin's work on Hilbert's Fifth Problem.

Since Montgomery is perhaps best known for his role in solving the conjecture known today as Hilbert's Fifth Problem, it is worth describing the relationship of the latter to the original question of Hilbert. In his well-known article Hilbert simply asked about the extent to which Lie's work on analytic transformation groups could be developed without assuming differentiability. Results of J. von Neumann [vN] confirmed that differentiability was not completely dispensable and, together with work of L. Pontrjagin [Pon], strongly suggested the following specialized version of Hilbert's problem:

(i) *If G is a topological group and a topological manifold, is G topologically isomorphic to a Lie group?*

This is the statement generally described as Hilbert's Fifth Problem. In studying this question one confronts the following more general problem fairly quickly:

(ii) *What is the topological structure of a locally compact (separable metric) topological group?*

This problem and the preceding one were solved by von Neumann [vN], Pontrjagin, and Chevalley for compact, abelian, and solvable (locally compact) groups respectively (bibliographic citations may be found in [50]). However, the general case remained undone until 1952 when Montgomery and Zippin supplied a positive answer to the first question and a general structure theorem for the second one modulo an assumption of finite dimensionality

[45-46]. Here is the main theorem of [45-46]:

Suppose G is a separable metric, locally compact, finite dimensional, connected, and locally connected topological group; furthermore, assume that all proper subgroups of G are generalized Lie groups[1]. Then G contains a closed, normal, generalized Lie subgroup H such that G/H is finite-dimensional and has no small subgroups.

Shortly afterwards H. Yamabe and M. Goto eliminated the need for a finite dimensionality restriction. The main result of [45-46], a deep theorem of A. Gleason [Gl], and a result of Goto combine to yield a positive answer to (i) and a solution to (ii).

The work of Montgomery and Zippin on Hilbert's Fifth Problem shows a clear pattern of movement from specific situations to the general case. The earlier papers on low dimensional groups contained ideas that figured importantly in the final assult in [45-46]. It is interesting to note the use of dimension theory as an important topological tool in these formative papers.

Refinements of the solutions to (i) and (ii) are due to Yamabe ([Yam 1], [Yam 2]; also see Kaplansky [Kap]). The results of [Yam 1] show that every locally compact group is a generalized Lie group in an appropriate sense (see the preceding footnote). A very striking application of (i) - (ii) due to M. Gromov characterizes the finitely generated groups that have polynomial growth functions [Gr].

Both Lie and Hilbert were interested in transformation groups of manifolds. Therefore the following question also fits naturally into the framework of Hilbert's original fifth problem:

Suppose the locally compact group G acts effectively on the manifold M. Is G a Lie group?

Results of Bochner and Montgomery provide a positive answer if the group acts by diffeomorphisms [26], and the result is also true in certain low-dimensional cases. Further information may be found in [Ray], [Yang], and [KKM] (the latter is in these proceedings). This is a very tantalizing problem that remains as elusive as ever today.

With the appearance of the book, "Topological Transformation Groups" [50], written with Zippin and published in 1955, the subject of transformation groups had come of age. In this book we find the basic terminology of the subject pretty much as we know it today. The book is devoted largely to a clear and complete exposition of the solutions to (i) and (ii). However, there are also two important chapters on transformation groups itself. These

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[1] A generalized Lie group is a topological group containing an open subgroup that is an inverse limit of Lie groups. See [45-46] for more information.

chapters present a large share of Montgomery's work up to that time in a unified framework, and this part of the book has had an important influence on subsequent developments. The book is still as interesting to read as it was in the nineteen fifties, and it is a valuable aid to understanding the motivating ideas behind much of present day research.

The work on Hilbert's Fifth Problem and other results in Montgomery's work showed that a large portion of the theory of transformation groups was intrinsically topological. Working from a more algebraic viewpoint, P. A. Smith had obtained fixed point theorems that carried the same message. During the middle and late nineteen fifties the viewpoints of Montgomery and Smith exerted strong influence on each other, and the result was a great flurry of activity in the subject. Innovative cohomological ideas were introduced to extend the work of Smith. Montgomery and his collaborators described many essential aspects of the orbit structure of a transformation group. These results on slices, principal orbits, and related concepts have become an indispensable part of the standard machinery of the subject ([52], [54], [55], [57], [61]; also see [Bor]). At the same time Montgomery and his collaborators studied a number of important types of group actions (e.g., see [52-54], [58-60], [62-67]). We shall say more later on the influence that some of these have had on the subject.

Although much of the theory of transformation groups is essentially topological, it was probably always clear that the theory of transformation groups could not be entirely topological. Montgomery and Zippin noted this in [47] and in pages 70-71 of [50]. Perhaps the most decisive example is Bing's nonsmoothable involution on S^3 with fixed point set S^2[Bi]; in this example the complement of the fixed point set has two components, each homeomorphic to the "inside region" of the Alexander horned sphere. During the late nineteen fifties and early nineteen sixties many other nonsmoothable group actions with highly nonstandard properties were constructed. These actions did not fit into any recognizable general framework. Furthermore, there were no encouraging signs to indicate that existing techniques would lead to an overall understanding. These limitations and the tremendous advances in differential topology during the same time led many workers in the area to concentrate on differentiable actions. The following quotation from page 3 of [Bor] reflects this viewpoint fairly well:

> It should also be pointed out that purely cohomological methods, while forming a major part of the subject at present, have their limitations, as is shown by well known counter examples; and that in view of this, it would certainly be very desirable to make more effective use of differentiability assumptions than has been possible so far.

In the mid nineteen sixties Montgomery and Yang embarked upon a significant program to study smooth actions on homotopy 7-spheres. This was a test for the newly created machinery of differential topology and surgery theory. The work of Montgomery and Yang showed that these techniques could be applied very directly and profitably to the study of differentiable group actions. During the past two decades surgery theory and transformation groups have enriched each other in many different ways.

Montgomery's influence has been pervasive. It is impressive to see how the papers of Montgomery and his collaborators have introduced striking new ideas that others have pursued, enriched with ideas of their own and built into entire theories. Here are a few specific examples:

(i) In [19] and [53] Montgomery, Samelson, and Yang showed that certain large group actions on spheres and Euclidean spaces were essentially unique. Results of the Hsiangs and their collaborators have placed these results into a broad general pattern; some of the extensive activity in this direction is discussed in [Hs] and [Sch 4] (in these proceedings).

(ii) The results of Montgomery and Yang on free and semifree smooth circle actions [71] were important in the development of general theories of semifree actions on homotopy spheres (e.g., see [Sch 2]). The later results of Montgomery and Yang on pseudofree circle actions ([76]; see also [DPS], Section 2, in these proceedings) were an important motivation for the equivariant surgery theory developed by T. Petrie (e.g., see [DPS], [Pet], [PR]).

(iii) The work of Conner and Montgomery on the symmetry properties of aspherical spaces [60] has led to the long, systematic study of actions on such manifolds by Raymond and his collaborators (e.g., see [CR], [LeR]).

(iv) The equivariant engulfing principle of Connell, Montgomery, and Yang [69] has proved to be fundamentally important in the study of differentiable actions that are topologically equivalent (compare [CaS], [DR], [Sch 1]).

(v) Several separate extensions of the slice and principal orbit theorems to proper, algebraic, and complex analytic actions have been obtained and applied to geometrical questions. References in this connection include several papers by R. Palais [Pa 2], D. Luna [Luna], and R. W. Richardson [Ric 1-3].

(vi) In [67] Conner and Montgomery constructed smooth actions of SO_3 on Euclidean spaces with no fixed points. A strong generalization of this due to the Hsiangs was fundamental to Oliver's original proof of the Conner Conjecture (see [O] or [Sch 3] in these proceedings).

(vii) In [79] Montgomery and Yang considered the combinatorial identities relating the dimensions of various fixed point sets obtained from a group action. This has led to a number of interesting results described elsewhere in these proceedings ([DPS], Section 4). The work of tom Dieck and Petrie [tDP] and tom Dieck (e.g., [tD]) also deals with questions of this sort.

(viii) In [22-24] and [26] Montgomery, first alone and later jointly with Bochner, proved several useful criteria for recognizing smooth and analytic actions of Lie groups. Such principles are often extermely valuable in topology and geometry; for example, it is important to know that the isometry group of a Riemannian manifold is a Lie group that acts smoothly. The theory of recognizing smooth actions has been developed extensively during the past few decades and it has been applied to many basic geometrical contexts. Further information may be found in [Pa 1], [ChKo], and Kobayashi's book [Kob]; for a relatively recent result see [Lo].

Montgomery has been at the Institute for Advanced Study in Princeton since 1948 and has held a professorship in the School of Mathematics since 1951 (emeritus since 1980). Prior to this he spent years at the Institute as a National Research Council Postdoctoral Fellow from 1934 to 1935, as a Guggenheim Fellow from 1941 to 1942, and as a member from 1945 to 1946 (during which time he worked in numerical analysis); he was also a visiting faculty member at Princeton University from 1943 to 1945. During much of the early part of his career Montgomery held a faculty position at Smith College; he went to Smith in 1935 after receiving his Ph. D. from the University of Iowa in 1933 under E. W. Chittenden (see [1]) and spending two years as a National Research Council Postdoctoral Fellow at Harvard and the Institute. In 1946 Montgomery joined the faculty at Yale and held this position until he went to the Institute in 1948. During his career Montgomery has been active and visible in professional organizations for mathematicians. Two indications of this are his terms as President of the American Mathematical Society in 1961 and 1962 and as President of the International Mathematical Union from 1975 to 1978. He has been a member of the National Academy of Sciences since 1955 and of the American Philosophical Society since 1958.

Concluding remarks

We have discussed Montgomery's publications at length and mentioned some of the positions he has held during his career. However, he has also made important contributions of a less tangible nature that deserve to be mentioned. Montgomery has also exerted a very profound impact on mathematics, and especially upon young mathematicians, in his capacity of Professor at the Institute for Advanced Study. He ran the topology seminar at the Institute for many years, and many of the great results of topology were first introduced in this seminar. Montgomery's leadership at the Institute ensured that the highest standards were maintained and yet everyone received a chance. More than anyone else he took the time to look after the visitors there and provide