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PREFACE

This book is a comprehensive text on linear algebra for students at the junior-senior level. Presumably, the students using this text will have some acquaintence with the basic matrix computations, although none is assumed. The book is divided into two parts. Part I treats the material which has by now become reasonably standard in most semester courses at this level. The main aim of Part I is to develop the material in as elementary a manner as possible, yet at the same time not compromise the development of the theory. Included are numerous examples and applications of the theory. In particular, applications which tie the material to other areas of mathematics are stressed.

Part II deals with advanced topics. Much of this material is more difficult and is designed for the students who continue with a second semester (or quarter). A major objective has been to write Part I at a level appropriate for students beginning a study of basic linear algebra with an emphasis on reasons why rather than on how to compute. The students who are successful with Part I should then be ready for the more advanced topics in Part II.

The book is written in the traditional mathematical style, with precisely formulated theorems followed by their proofs. Every instructor of upper-division mathematics understands the difficulties of the transition from computation-oriented classes to a class of this type. This text was written to help students over this obstacle. It offers no magic ladder to climb the barrier, nor do any technical gimmicks (computer programs, elaborate graphics, etc.) accompany this text. Instead the book begins with concrete and familiar concepts and then develops the theory at a pace that allows students time to become accustomed to the style. The foundations of linear algebra come from the study of systems of equations. For this reason this text, as much as possible, utilizes (and interprets) gaussian elimination in the proofs of the results in the earlier chapters. As the book develops the level of abstaction is increased so that by the time Part II is reached the invariant point of view is firmly entrenched.

The material in the text is developed without gaps; that is, no details of results are left to the reader or as exercises. At the same time the ideas are informally discussed and many of the common pitfalls for new students pointed out. The author believes that mathematical proof is an appropriate form of communication for upper-division university students and if we expect them to learn to speak this language, then they need to be provided with a complete and accurate model to follow. Most sections end with at least twice as many problems as need to be assigned. These problems include both the standard computational exercises as well as theoretical questions of varying difficulty. Answers and hints to most odd-numbered problems can be found at the end of the text.

There are many levels upon which one learns mathematics. Initially one learns how to perform certain computations. Then one studies examples of such calculations, searching for patterns and common themes. The conclusions of such observations are formulated as precisely as possible, and finally the mathematician manipulates the concepts into a logical order and proves results from basic principles. Students using this text will experience these different levels. In addition to learning the theory, one purpose of the "show" and "prove" problems in this text is for students to develop clear communication skills. This text acquaints students with the language of modern mathematics and at the same time exposes them to the process of transforming a computational subject into a conceptual framework with much greater applicability.

Uses of the Text

There is considerable flexibility in the design of a course using this text. Part I develops the material found in most one-semester linear algebra courses at the junior-senior level. This half of the text is designed primarily for such courses. It should be serviceable for students with a variety of backgrounds who need a course that gives more than computations. Part II develops a variety of advanced topics and applications. Depending upon students' backgrounds and interests, material from the second part can be incorporated into the latter weeks of a semester course. Taken together, both parts contain sufficient material for a rigorous full-year course.

It is not possible, nor reasonable, that the details of every proof in the text be covered in class. Instructors will need to devote time to discussing the intuitive ideas behind the results, as well as to developing the examples and discussing problems. A large amount of the feeling of this subject is best acquired verbally through classroom interaction. A major goal in writing this book has been to provide a text from which the instructor can feel comfortable about sending the students to dig out details as well as to find additional examples. The following chapter-by-chapter discussion indicates how the material is organized.

Part I

Chapter 0: Chap. 0 is devoted to matrix algebra and the basic theory of systems of linear equations. Students will have varying degrees of familiarity with these results from previous courses. Consequently, the amount of time devoted to these topics will vary anywhere from 0 to 5 or 6 (hour) lectures. The chapter develops the foundations of the subject and at the same time introduces the student to the rigor and style of the text in a familiar setting. Sec. 0.7 provides a quick review of the material needed in the sequel and is written for those who would like to cover the material of this chapter in a a single lecture.

Throughout Chap. 0, the term "field" is used to denote either the real or complex numbers. This has been done because many instructors prefer to introduce this material in as familiar a setting as possible, that is, over **R** and **C**. However, other instructors prefer to discuss general fields immediately. In this case, because the term "field" is used, all the results and their proofs in Chap. 0 can be interpreted in the general setting. This flexibility allows these instructors to give the definition of a field (Definition 1.1.1) at the beginning of Chap. 0.

Chapter 1: Chap. 1 presents the basic theory of vector spaces. Vector spaces are introduced as subspaces of $\mathbf{F}^p$ ($\mathbf{F}$ is a field). The concepts of linear independence and span are introduced concretely in this setting and the relationship between these concepts and matrix algebra is explored. The text then makes the transition to the general setting and defines arbitrary vector spaces. The traditional topics of bases, dimension, and coordinates for finite-dimensional vector spaces are developed in detail. The last (optional) section is an introduction to algebraic codes over $\mathbf{Z}/2\mathbf{Z}$, which students enjoy as an initial application of the abstract theory.

Chapter 2: Chap. 2 is devoted to a complete development of the theory of determinants. The material is independent from Chap. 1 and, if the instructor chooses, can be covered immediately after Chap. 0. All the basic results from the theory of determinants are rigorously proved. Most instructors will choose to omit many of the details and will instead explain the ideas and concentrate on the applications developed in Sec. 2.3. The last Sec. 2.4 is a listing of the key results about determinants needed for the sequel. It can be used by those instructors who desire only to give a quick review of the properties of determinants without dealing with the theory.

Chapter 3: Chap. 3 treats the basic theory of linear transformations. Matrix representations for linear transformations and matrix techniques are developed thoroughly. In addition, numerous non-matrix examples are studied in order to illustrate the need for a general theory. Eigenvalues and eigenvectors are developed in Sec. 3.4 and their connection with diagonalizability in the section that follows. As applications, Sec. 3.6 gives informal discussions of Markov chains, difference equations, and Gerschgorin's theorem.

Chapter 4: The basic theory of norms, inner products, and orthogonality are developed here. Direct sums and orthogonal projections are introduced. This chapter has a more applied flavor in that the ideas behind least-squares problems are developed in detail. In addition the basic results from spectral theory and polar decomposition are proved. The final section is devoted to optimal least-squares problems and the pseudoinverse.

Part II

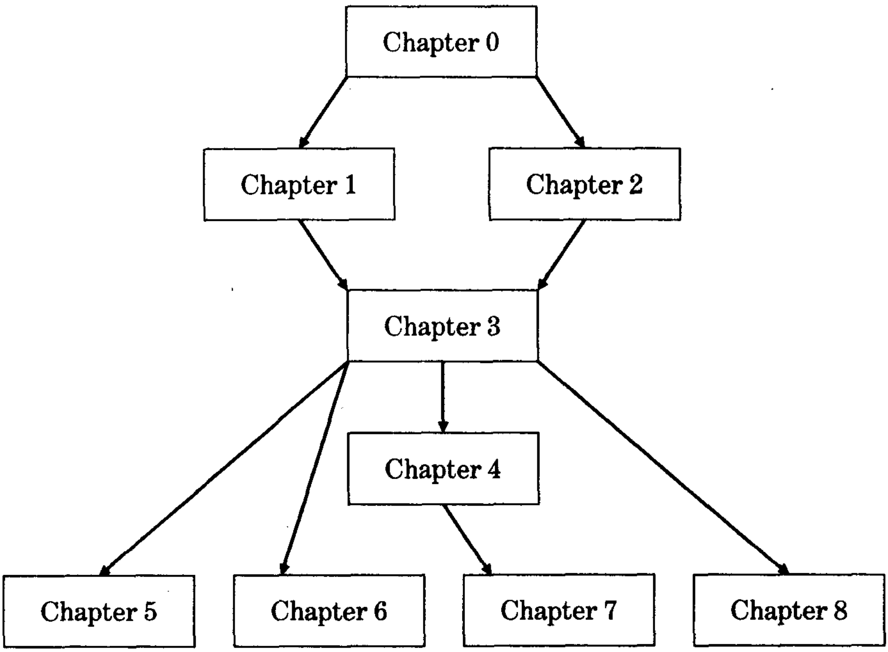
Chapter 5: Chap. 5 treats the equivalence of the concepts of symmetric bilinear forms, symmetric matrices, and quadratic forms (in characteristic different from 2). The diagonalizability of quadratic forms and the signature of real quadratic forms are major topics. The connections with advanced calculus are described.

Chapter 6: The level of difficulty of the material increases throughout this chapter; the material becomes conceptually the most difficult in the text. The theory of a single linear operator on a finite-dimensional vector space is studied in detail with the main goal the development of the rational and Jordan canonical forms. The chapter begins with the detailed study of examples, so that the basic ideas behind the canonical forms can be understood before the difficult existence theorems are tackled. In fact, many users of the text may not have the time (or the desire) to give the existence proofs. It is a profitable experience for students to see some of the material of Secs. 6.1 to 6.7, even though the proofs are not given. This chapter has been written in such a way as to make this approach work. The last sections are theoretical and are tough even for the best students. The Smith normal form, which gives students an effective algorithm for computing the rational form, is developed in Sec. 6.8.

Chapter 7: In this chapter the style changes considerably. The purpose of this chapter is to expose students in an informal way to some of the infinite-dimensional topics. No attempt is made to be comprehensive, and the level of rigor varies considerably. The sections on the Wronskian and on systems of linear differential equations are important classroom topics and can be covered once the material from Chap. 3 is mastered. The remainder of this chapter requires the material from Chap. 4. It consists of an informal discussion of orthogonal polynomials, Fourier series, and the proof of the existence of a basis for an arbitrary vector space. These sections are written for students to read and increase their horizons (and are not suitable for extensive classroom development). Hopefully, the students will find reading them enjoyable and profitable.

Chapter 8: The only prerequisite for reading this chapter is the material of Chap. 3; however knowledge of multivariable calculus is necessary in order for the last section to be meaningful. The first two sections deal with multilinear algebra and tensors. The instructor who likes to discuss the double dual will find it in Sec. 8.1. As with Chap. 7, these sections are written with much less formality. The final section is basically an essay whose goal is to help the student connect the ideas developed earlier (namely tensors) with the basic concepts of multivariable calculus and advanced geometry.

The following diagram indicates the interdependence of the chapters in this book.



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Bill Jacob
Berkeley, California
June, 1989

CONTENTS

Part II: Advanced Topics

Part I: Basic Linear Algebra

Chapter 0

Linear Equations and Matrices

0.1 Systems of Linear Equations

This chapter develops the theory of systems of linear equations. Matrices and matrix operations are introduced. So are the concepts of matrix rank and matrix inverses. The main task of this chapter is to lay rigorously the foundations for the rest of the book. While reading this chapter, students should keep in mind that it is not enough to understand merely the computations. What is crucial is *why* things work the way they do. If the reasons are carefully digested, then the student can master the basic concepts of linear algebra introduced in Chap. 1.

Linear algebra begins with the study of systems of linear equations. The foundations of this subject should be familiar from high school algebra, where it is usual to study systems of two or three equations. This first section treats a far more general situation, where the numbers of equations and variables are arbitrary and *not necessarily the same.* Such generality is critical for the development of the subject of linear algebra. In order to solve general systems of equations, one can use the computational algorithm known as gaussian elimination. The ideas behind this algorithm are extremely simple, yet its consequences are powerful. This important tool will be used explicitly (or implicitly) throughout the remainder of the book.