MATHEMATICS OF FINANCE

SECOND EDITION

ZIMA-BROWN

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McGraw-Hill Ryerson Limited
Toronto Montreal New York St. Louis San Francisco Auckland Bogotá
Guatemala Hamburg Johannesburg Lisbon London Madrid Mexico
New Delhi Panama Paris San Juan São Paulo Singapore Sydney Tokyo

MATHEMATICS OF FINANCE,
Second Edition

ISBN: 0-07-548490-0

1 2 3 4 5 6 7 8 9 0 HR 4 5 6 7 8 9 0 1 2 3

Printed and bound in Canada

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Canadian Cataloguing in Publication Data

Zima, Petr, date
Mathematics of finance

Includes index.
ISBN 0-07-548490-0

I. Business mathematics. I. Brown, Robert L., date
II. Title.

HF5691.Z55 1982 513'.93 C82-094993-0

PREFACE

Knowledge in the field of the mathematics of finance continues to be of utmost importance to the well qualified high school, community college, or university student. Recent high and rapidly changing rates of interest have only reinforced that fact.

In the world of finance, it is extremely important that one has an up-to-date text written in a manner that will be easily understood by both instructor and student.

Our new text has been written with these needs in mind. In particular, it has been written assuming that the student will be using a pocket calculator with a full range of functions including a logarithmic function and a power function. With this in mind, and because interest rates can change so rapidly and so significantly, we have excluded the interest rate tables included in more traditional texts. Through experience in our own classrooms, we have found that this deletion is advantageous to the student who must solve complex problems.

This book is designed to be used by students in the last year of high school, by business and business administration students at the community college level, and by business administration and economics students taking introductory courses at the university level. We have assumed some mathematical background in the students using this text. A wide mathematical background, however, is not a prerequisite. In particular, we have not included introductory chapters on basic algebraic techniques. But we have included four appendices with exercises that deal in detail with the topics of exponents and logarithms, progressions, linear interpolation and continuous compounding. We have tried to use as many examples as possible and have included both basic problems in PART A of the Exercises which will help students with the learning of basic concepts, together with advanced problems in PART B of the Exercises which will sharpen high-level skills. At the end of each chapter we included Review Exercises.

The numeric examples in the text have been SI metricated. That is, numbers are in their standard international form as now required by the Ministry of Education in several provinces.

We are indebted to the many people whose constructive criticism resulted in improvements in the original text. Despite a careful scrutiny, it is inevitable that flaws will remain. For these, the authors accept full responsibility and welcome any suggestions. We would like to thank Lynda Hohner for the fine work she performed in typing this manuscript.

We feel that this textbook will answer a growing need for an authentic and up-to-date Canadian text on the applications of the Mathematics of Finance.

PETR ZIMA
ROBERT L. BROWN

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Simple Interest and Simple Discount

1.1 Simple Interest

Suppose that an investor lends money to a debtor. The debtor must pay back the money originally borrowed, and also the fee charged for the use of the money, called **interest**. From the investor's point of view, interest is income from invested capital. The capital originally invested in an interest transaction is called **the principal**. The sum of the principal and the interest due is called the **amount** or **accumulated value**. Any interest transaction can be described by the **rate of interest**, which is the ratio of the interest earned in one time unit to the principal.

In early times, the principal lent and the interest paid might be tangible goods (e.g. grain). Now, they are most commonly in the form of money. The practice of charging interest is as old as the earliest written records of mankind. Four thousand years ago, the laws of Babylon referred to interest payments on debts.

At **simple interest**, the interest is computed on the original principal during the whole time, or term of the loan, at the stated annual rate of interest.

We shall use the following notation:

P = the principal, or the present value of S, or the discounted value of S, or the proceeds.

I = simple interest.
S = the amount, or the accumulated value of P, or the maturity value of P.
r = rate of interest per year.
t = time in years.

Simple interest is calculated by means of the formula

$$\boldsymbol{I = Prt} \tag{1}$$

From the definition of the amount S we have

$$S = P + I$$

By substituting for $I = Prt$ we obtain S in terms of P, r, and t

$$S = P + Prt$$

$$\boldsymbol{S = P(1 + rt)} \tag{2}$$

The factor $(1 + rt)$ in formula (2) is called an **accumulation factor at simple interest** and the process of calculating S from P by formula (2) is called **accumulation at simple interest.** From formula (2) we can express P in terms of S, r, and t and obtain

$$\boldsymbol{P = \frac{S}{1 + rt} = S(1 + rt)^{-1}} \tag{3}$$

When we calculate P from S, we call P the present value of S or the discounted value of S. The factor $(1 + rt)^{-1}$ in formula (3) is called a **discount factor at simple interest** and the process of calculating P from S is called **discounting at simple interest**, or simple discount at an interest rate.

The time t must be in years. When the time is given in months, then

$$t = \frac{\text{number of months}}{12}$$

When the time is given in days, there are two different varieties of simple interest in use:

1. **Exact interest**, where $t = \dfrac{\text{number of days}}{365}$

 i.e., the year is taken as 365 days (leap year or not).

2. **Ordinary interest**, where $t = \dfrac{\text{number of days}}{360}$

 i.e., the year is taken as 360 days.

The general practice in Canada is to use exact interest, whereas the general practice in the United States and in international business transactions is to use ordinary interest. In this textbook exact interest is used all the time unless specified otherwise.

Example 1 Find the exact and ordinary simple interest on a 90-day loan of \$500 at $8\frac{1}{2}$%.

Solution We have $P = 500$, $r = 0.085$, time = 90 days

$$\text{Exact interest} = 500 \times 0.085 \times \frac{90}{365} = \$10.48$$

$$\text{Ordinary interest} = 500 \times 0.085 \times \frac{90}{360} = \$10.63$$

Notice that ordinary interest is always greater than the exact interest and thus it brings increased revenue to the lender.

Example 2 A couple borrows $10 000. The annual interest rate is $10\frac{1}{2}\%$, payable monthly, and the monthly payment is $200. How much of the first payment goes to interest and how much to principal?

Solution We have $P = 10\,000$, $r = 0.105$, $t = \frac{1}{12}$, and

$$I = 10\,000 \times 0.105 \times \tfrac{1}{12} = \$87.50$$

The interest for the first month is $87.50 and $112.50 is applied to principal reduction.

Example 3 A loan shark made a loan of $100 to be repaid with $120 at the end of one month. What was the annual interest rate?

Solution We have $P = 100$, $I = 20$, $t = \frac{1}{12}$, and

$$r = \frac{I}{Pt} = \frac{20}{100 \times \frac{1}{12}} = 240\%$$

Example 4 Sixty days after borrowing money a person pays back exactly $200. How much was borrowed if the $200 payment includes the principal and simple interest at 9%?

Solution We have $S = 200$, $r = 0.09$, and $t = \frac{60}{365}$. Substituting in formula (3) gives

$$P = \frac{200}{1 + 0.09(\frac{60}{365})} = \$197.08$$

Example 5 How long will it take $3000 to earn $60 interest at 6%?

Solution We have $P = 3000$, $I = 60$, $r = 0.06$, and

$$t = \frac{I}{Pr} = \frac{60}{3000 \times 0.06} = \frac{1}{3} = 4 \text{ months}$$

Example 6 *Cash discounts on purchase of merchandise.* To encourage prompt payments of invoices many manufacturers and wholesalers offer cash discounts for payments in advance of the final due date. The following typical credit terms may be printed on sales invoices:

> 2/10, *n*/30—Goods billed on this basis are subject to a cash discount of 2% if paid within ten days. Otherwise, the full amount must be paid not later than thirty days from the date of the invoice.

A buyer who takes advantage of cash discounts in effect lends money to the seller and receives as interest the cash discount. Interest rates earned in this manner are usually very high. The following example illustrates the use of a cash discount.

Example: A merchant receives an invoice for a motor boat for $4000 with terms 4/30, *n*/100. What is the highest simple interest rate at which he can afford to borrow money in order to take advantage of the discount?

Solution Suppose that the merchant will take advantage of the cash discount of 4% of 4000 = $160 by paying the bill within 30 days from the date of invoice. He needs to borrow 4000 − 160 = $3840 for 70 days. The interest he should be willing to pay on borrowed money should not exceed the cash discount $160.

We have $P = 3840$, $I = 160$, $t = \frac{70}{365}$, and we calculate

$$r = \frac{I}{Pt} = \frac{160}{3840 \times \frac{70}{365}} = 21.73\%$$

The highest simple interest rate at which the merchant can afford to borrow money is 21.73%. If he can borrow money, say at rate 15%, he should do so and realize a profit equal to the difference between the cash discount of $160 and the interest he must pay on a 70-day loan of $3840 at rate 15%. Interest at 15% on $3840 for 70 days $= 3840 \times 0.15 \times \frac{70}{365} = 110.47$. Thus, his profit on the transaction would be $160 - 110.47 = \$49.53$.

Exercise 1.1

1. Find the accumulated value of $500 at 11% ordinary simple interest over 60 days.
2. At what simple rate of interest will $1000 accumulate to $1420 in $2\frac{1}{2}$ years?
3. How long will it take $500 to accumulate to $560 at 12% simple interest?
4. Find the ordinary and exact simple interest on $5000 for 90 days at $10\frac{1}{2}\%$.
5. A student lends his friend $10 for one month. At the end of the month he asks for repayment of the $10 plus purchase of a chocolate bar worth 50¢. What ordinary simple interest rate is implied?
6. What principal will accumulate to $5100 in 6 months if the rate is 9%?
7. What principal will accumulate to $580 in 120 days at 18% exact simple interest?
8. Find the accumulated value of $1000 over 65 days at $11\frac{1}{2}\%$ using both ordinary and exact simple interest.
9. Find the interest earned on $600 over 118 days at 16% using both ordinary and exact simple interest.
10. A man borrows $1000 for 220 days at 17%. What amount must he repay?
11. Find the discounted value of $500 over 82 days at 9% using both ordinary and exact simple interest.
12. Find the discounted value of $100 due in 3 months if the rate is 11%.
13. A bank pays 10% per annum on savings accounts. Interest is credited quarterly on March 31, June 30, September 30, and December 31 based on the minimum quarterly balance. If a person opens an account with a deposit of $200 on January 1 and withdraws $100 on August 8, how much interest is earned in the first year?
14. Mr. X has a Special Savings Account which pays interest at 12% per annum. Interest is calculated by the bank on the minimum monthly balance and is paid into the account on December 31. Given the following transactions for the account opened January 1, find the interest earned in the first year.

Date	Deposits	Withdrawals	Balance
January 1	$100		$100
February 3	$200		$300
April 14		$150	$150
May 18	$300		$450
July 7		$200	$250
September 15		$150	$100
November 3	$100		$200

15. A cash discount of 2% is given if a bill is paid 20 days in advance of its due date. At what interest rate could you afford to borrow money to take advantage of this discount?
16. A merchant receives an invoice for \$1000 with terms 3/10, *n*/60. If he pays on the 10th day what rate of interest does he earn?
17. A merchant receives an invoice for \$2000 with terms 2/20, *n*/60. What is the highest simple interest rate at which he can afford to borrow money in order to take advantage of the discount?
18. The ABC general store receives an invoice for goods totalling \$500. The terms were 3/10, *n*/30. If the store were to borrow the money to pay the bill in 10 days what is the highest interest rate at which the store can afford to borrow?

1.2 The Time between Dates

There are two ways to calculate the number of days between calendar dates.

The most common method is to calculate the exact number of days including all days except the first. The time computed in this way is called the **exact time**. A simple way to determine the exact number of days is to use Table I (see page 247). Table I is essentially a calendar, which gives the serial numbers of the days in the year. The exact time is obtained as a difference between serial numbers of the given dates. In leap years, the serial number of the day is increased by 1 for all dates after February 28.

Another method is based on the assumption that all full months contain 30 days. The time computed this way is called the **approximate time**.

Example 1 Find the exact and approximate time between March 15 and September 3.

Solution From Table I, March 15 is the 74th day of the year and September 3 is the 246th day of the year. The exact time is $246 - 74 = 172$ days.

For the approximate time we arrange the data in the table shown below:

Date	Month	Day	Month	Day
September 3	9	3	8	33
March 15			3	15
Difference			5	18

The approximate time is 5 months and 18 days, or 168 days.

When the time is given indirectly as the time between dates, we can use either exact or approximate time and compute either exact or ordinary simple interest. Thus there are four distinct ways to compute simple interest between dates, using

1. Exact time and exact interest.
2. Exact time and ordinary interest.

3. Approximate time and exact interest.
4. Approximate time and ordinary interest.

The general practice in Canada is to use Method 1, i.e., exact time and exact interest, in all simple interest calculations. In this book we use Method 1 unless otherwise specified.

Method 2 is also known as the **Banker's Rule** and is used widely in business practice in the United States and in international business transactions. Methods 3 and 4 are used very rarely.

Example 2 On November 3, 1978 a man borrowed $500 at 9%. The debt is repaid on February 8, 1979. Find the simple interest using the four methods.

Solution First we calculate exact and approximate time. From Table I, February 8 is the 39th day of the year and November 3 is the 307th day of the year.

February 8, 1979:	39 + 365 =	404th day, counting from January 1, 1978
November 3, 1978:		307th day, counting from January 1, 1978
Exact time between dates		97 days

For the approximate time we arrange the data in the table shown below, on the basis of months starting from January 1, 1978.

Date	Month	Day
February 8, 1979	14	8
November 3, 1978	11	3
Difference	3	5

The approximate time is 3 months and 5 days, or 95 days.

Exact time and exact interest $I = 500 \times 0.09 \times \frac{97}{365} = \$11.96$

Exact time and ordinary interest $I = 500 \times 0.09 \times \frac{97}{360} = \$12.13$

Approximate time and exact interest $I = 500 \times 0.09 \times \frac{95}{365} = \$11.71$

Approximate time and ordinary interest $I = 500 \times 0.09 \times \frac{95}{360} = \$11.88$

Notice the differences in simple interest depending on the method used. This brings out the fact that in computing simple interest, as in all problems in the mathematics of finance both parties to the transaction should understand what method is to be used. The most favorable method for the creditor is the Banker's Rule (exact time and ordinary interest), as it usually yields the maximum interest. (This would not be true, for example, for the time interval February 4 to March 2 of any year, where the approximate time is greater than the exact time.)

Exercise 1.2

1. Find the exact and approximate time from April 18 to November 3.
2. Find the exact and approximate time from October 2 to June 15.
3. On April 7, 1978, Mr. X borrows $1000 at 8%. He repays the debt on November 22, 1978. Find the simple interest using the four methods.
4. A sum of $2000 is invested from May 18, 1982 to April 8, 1983 at 16% simple interest. Find the amount of interest earned using the four methods.
5. On January 1, Mr. A borrows $1000 on a demand loan from his bank. Interest is paid at the end of each quarter (March 31, June 30, September 30, December 31) and at the time of the last payment. Interest is calculated at the rate of 12% on the balance of the loan outstanding. Mr. A repaid the loan with the following payments:

March 1	$100
April 17	$300
July 12	$200
August 20	$100
October 18	$300
	$1000

   Calculate the interest payments required and the total interest paid. (Follow normal Canadian practice.)
6. Find the total interest paid in problem 5 using the Banker's Rule.

1.3 Equations of Value

All financial decisions must take into account the basic idea that *money has time value*. In a financial transaction involving money due on different dates, every sum of money should have an attached date, the date on which it falls due. That is, the mathematics of finance deals with *dated values*. This is one of the most important facts in the mathematics of finance.

Illustration: At a simple interest rate of 8%, $100 due in 1 year is considered to be equivalent to $108 due in 2 years since $100 would accumulate to $108 in 1 year. In the same way

$$100(1 + 0.08)^{-1} = \$92.59$$

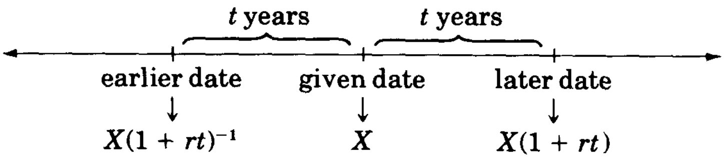
would be considered an equivalent sum at present.

In general, we compare dated values by the following **definition of equivalence**:

$\$X$ due on a given date is equivalent at a given simple interest rate r to $\$Y$ due t years later if

$$Y = X(1 + rt) \quad \text{or} \quad X = \frac{Y}{1 + rt} = Y(1 + rt)^{-1}$$

The following time diagram illustrates dated values equivalent to a given dated value X.



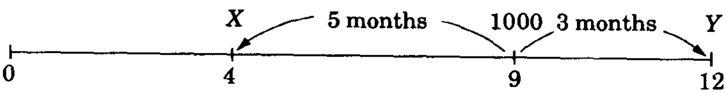
Note Based on the time diagram above we can state the following simple rules:

When we move money forward, we accumulate, i.e., multiply the sum by an accumulation factor $(1 + rt)$

When we move money backward, we discount, i.e., multiply the sum by a discount factor $(1 + rt)^{-1}$

Example 1 A debt of \$1000 is due at the end of 9 months. Find an equivalent debt at a simple interest rate of 9% at the end of 4 months and at the end of 1 year.

Solution Let us arrange the data on a time diagram below.



According to the definition of equivalence

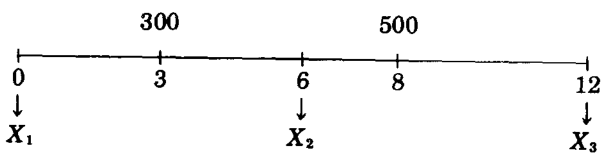
$$X = 1000[1 + (0.09)(\tfrac{5}{12})]^{-1} = \$963.86$$

$$Y = 1000[1 + (0.09)(\tfrac{3}{12})] = \$1022.50$$

The sum of a set of dated values, due on different dates, has no meaning. We have to replace all the dated values by equivalent dated values, due on the same date. The sum of the equivalent values is called the **dated value of the set**.

Example 2 A person owes \$300 due in 3 months and \$500 due in 8 months. What single payment (a) now; (b) in 6 months; (c) in 1 year, will liquidate these obligations if money is worth 8%?

Solution



We calculate equivalent dated values of both obligations at the three different times and arrange in the table below.

Obligations	Now	In 6 months	In 1 year
First	294.12	306.00	318.00
Second	474.68	493.42	513.33
Sum	X_1 = 768.80	X_2 = 799.42	X_3 = 831.33

One of the most important problems in the mathematics of finance is the replacing of a given set of payments by an equivalent set.

We say that two sets of payments are equivalent at a given simple interest rate if the dated values of the sets, on any common date, are equal. An equation stating that the dated values, on a common date, of two sets of payments are equal is called an **equation of value** or an **equation of equivalence**. The date used is called the **focal date** or the **comparison date**.

A very effective way to solve many problems in mathematics of finance is to use the equation of value. The procedure is carried out in the following steps.

Step 1 Make a good time diagram, showing the dated values of obligations on one side of the time line and the dated values of payments on the other side. A good time diagram is of great help in the analysis and solution of problems.

Step 2 Select a focal date and bring all the dated values to the focal date using the specified interest rate.

Step 3 Set up an equation of value at the focal date.

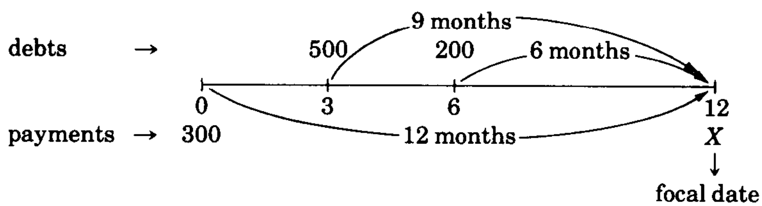
Step 4 Solve the equation of value using methods of algebra.

In simple interest problems the answer will vary slightly with the location of the focal date. It is therefore important that the parties involved in the financial transaction agree on the location of the focal date.

In compound interest (Chapter 2) any date may be used as a focal date; that is, the answer is independent of the location of the focal date.

Example 3 A debtor owes \$500 due in 3 months and \$200 due in 6 months. If his creditor accepts \$300 now, how much will be required to liquidate the two obligations at the end of 1 year, provided they agree to use an interest rate of 10% and a focal date at the end of 1 year.

Solution We arrange all the dated values on a time diagram.



Equation of value at the end of 12 months:

$$\text{dated value of the payments} = \text{dated value of the debts}$$
$$X + 300[1 + (0.10)(\tfrac{12}{12})] = 500[1 + (0.10)(\tfrac{9}{12})] + 200[1 + (0.10)(\tfrac{6}{12})]$$
$$X + 330.00 = 537.50 + 210.00$$
$$X = \$417.50$$

The payment of \$417.50 will be required at the end of 1 year to liquidate both obligations.

Example 4 A person borrows \$1000 at 11%. He is to repay the debt with 3 equal payments, the first at the end of 3 months, the second at the end of 6 months and the third at the end of 9 months. Find the size of the payments. Put the focal date (a) at the present; (b) at the end of 9 months.