STATISTICAL FIELD THEORY

Volume 1

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Preface

Some ten years ago, when completing with J.-B. Zuber a previous text on *Quantum Field Theory*, the senior author was painfully aware that little mention was made that methods in statistical physics and Euclidean field theory were coming closer and closer, with common tools based on the use of path integrals and the renormalization group giving insights on global structures. It was partly to fill this gap that the present book was undertaken. Alas, over the five years that it took to come to life, both subjects have undergone a new evolution. Disordered media, growth patterns, complex dynamical systems or spin glasses are among the new important topics in statistical mechanics, while superstring theory has turned to the study of extended systems, Kaluza–Klein theories in higher dimensions, anticommuting coordinates ... in an attempt to formulate a unified model including all known interactions. New and sophisticated techniques have invaded statistical physics, ranging from algebraic methods in integrable systems to fractal sets or random surfaces. Powerful computers or special devices provide "experimental" means for a new brand of theoretical physicists. In quantum field theory, applications of differential topology, geometry, Riemannian manifolds, operator theory ... require a deeper background in mathematics and a knowledge of some of its most recent developments. As a result, when surveying what has been included in the present volume in an attempt to uncover the basic unity of these subjects, the authors have the same unsatisfactory feeling of not being able to bring the reader really up to date. It is presumably the fate of such endeavours to always come short of accomplishing their purpose.

With these shortcomings fully admitted, we have tried to present to the reader an overview of the main themes which justify the title "Statistical field theory." This interpretation of Euclidean field theory offers a new language, effective computing means, as well as a natural and consistent short-distance cutoff. In other words, it allows one to give a global meaning to path integrals, to discover possible anomalies arising from integration measures, or to understand in simple terms systems with redundant variables such as gauge models. The theory of continuous phase transitions provides a bridge between probabilistic mechanics and continuous field theory, using the renormalization group to filter out relevant operators and interactions. Many authors contributed to these views, culminating in the work of K. Wilson and his collaborators and followers, which promoted the renormalization group as a universal tool to analyse the large-distance behaviour. It still retains its value, while new developments take place, particularly with conformal, or local scale invariance coming to prominence in the study of two-dimensional systems.

The content of this book is naturally divided into two parts. The first six chapters describe in succession Brownian motion, its anticommutative counterpart in the guise of Onsager's solution to the two-dimensional Ising model, the mean-field or Landau approximation, scaling ideas exemplified by the Kosterlitz–Thouless theory for the XY-transition, the continuous renormalization group applied to the standard φ^4 theory, the simplest typical case, and lattice gauge theory as an attempt to understand quark confinement in chromodynamics.

The next five chapters (in volume 2) cover more diverse subjects. We give an introduction to strong coupling expansions and various means of analyzing them. We then briefly introduce Monte Carlo simulations with an emphasis on the applications to gauge theories. Next we turn to the significant advances in two-dimensional conformal field theory, with a lengthy presentation of the methods as well as early results. A chapter on simple disordered systems includes sample applications of fermionic techniques with no pretence at completeness. The final chapter is devoted to random geometry and an introduction to the Polyakov model of random surfaces which illustrates the relations between string theory and statistical physics.

At the price of being perhaps a bit repetitive, we have tried in the first part to introduce the subject in an elementary fashion. It is, however, assumed that the reader has some familiarity with thermodynamics as well as with quantum field theory. We often switch from one to the other interpretation, assuming that it will not be disturbing once it is realized that the exponential of the action plays the role of the Boltzmann–Gibbs statistical weight. The last chapters cover subjects still in fast evolution.

Many important subjects could unfortunately not be covered. In random order they include dynamical critical phenomena, renormalization of σ-models or non-Abelian gauge fields except for a mention of lowest order results, topological aspects, classical solutions, instantons, monopoles, anomalies (except for the conformal one). Integrable systems are missing apart from the two-dimensional Ising model. Quantum gravity *à la Regge* is only mentioned. The list could, of course, be made much longer. Our involvement in some of the topics has certainly produced obvious biases and overemphases in certain instances. We have tried, as much as possible, to correct for these defects as well as for the numerous omissions by including at the end of each chapter a section entitled "Notes." Here we quote our sources, original articles, reviews, books and complementary material. These notes are purposely scattered through the volume, as we are sure that our quotations are very incomplete. A fair bibliography in such a large domain is beyond human capacities. Should any one feel that his or her work has not been reported or not properly mentioned, he or she is certainly right and we present our most sincere apologies. On the other hand we did not hesitate to use and sometimes follow very closely some articles or reviews which served our purpose. For instance chapter 5 is built around the definitive contributions of E. Brézin, J.-C. Le Guillou, J. Zinn-Justin and G. Parisi. Except for some further elaboration by the authors themselves, it was futile to try to improve on their work. Further examples are mentioned in the notes. It is the very nature of a survey such as this one to be inspired largely by other people's works. We hope that we did not distort or caricature them.

A book might give the illusion, especially to students, that some knowledge has become definitive and that the authors understand every part of it. This is a completely false view. No one can really fully master even his own subject, and this is luckily a source of progress. It is in the process of learning, of objecting, of finding misprints and errors, in rediscovering for oneself, that one gets the real benefits. It is very likely that, in spite of our care, many errors have crept in here and there. We welcome gladly comments and criticisms.

It was very hard to keep uniform notation throughout the text, even sometimes in the same chapter. This is a standard difficulty, especially when traditional notation in a given domain comes into conflict with those used in another one, and a compromise is necessary. We hope that this will not be a source of confusion for the reader.

We have added appendices which generally gather material in very concise form. They should be supplemented by further reading. For instance appendix C of chapter 9 is obviously insufficient to describe finite and infinite Lie algebras and their representations. This appendix is, rather, meant to induce the interested reader to study the subject further. This is also the nature of several sections where the degree of mathematical sophistication seems to increase beyond the standard background, reflecting recent trends. It was felt difficult to omit these developments but also impossible to give a proper complete introduction.

Included in small type here and there are comments, exercises and short complements ... It was felt inappropriate to develop a scholarly set of problems. In this respect the whole text can be read as a problem book.

One of the "heroes" of the whole subject of statistical physics, in one guise or another, is still to this day our old friend the Ising model. We keep a few bottles of good old French wine for the lucky person who solves it in three dimensions. It would seem appropriate to create in the theoretical physics community a prize for its solution, analogous to the one founded at the beginning of the century for the proof of Fermat's theorem. Both subjects have a similar flavour, being elementary to formulate. While it is to be presumed that the answer itself is to a large extent inessential, they motivated creative efforts (and still do) which go largely beyond the goal of solving the problem itself.

Among the many books which either overlap or amply complement the present one, the foremost are of course those in the series edited by C. Domb and M.S. Green and now J. Lebowitz, entitled *Phase transitions and critical phenomena* and published through the years by Academic Press (New York). We freely refer to this series in the notes. Let us also quote here a few others, again without pretence at exhaustivity. On the statistical side, K. Huang, *Statistical mechanics*, Wiley, New York (1963); H.E. Stanley, *In-*

troduction to phase transitions and critical phenomena, Oxford University Press (1971); S.K. Ma, *Modern theory of critical phenomena*, Benjamin, New York (1976) and *Statistical mechanics*, World Scientific, Singapore (1985); D.J. Amit, *Field theory, the renormalization group and critical phenomena*, 2nd edition, World Scientific, Singapore (1984).

Books on the path integral approach to field theory are by now numerous. Among them, the classical one is R.P. Feynman and A.R. Hibbs, *Quantum mechanics and path integrals*, McGraw Hill, New York (1965). Further aspects are covered in C. Itzykson and J.-B. Zuber, *Quantum field theory*, McGraw Hill, New York (1980); P. Ramond, *Field theory, a modern primer*, Benjamin/Cummings, Reading, Mass. (1981); J. Glimm and A. Jaffe, *Quantum physics*, Springer, New York (1981). To fill some gaps on other developments in field theory, see S. Coleman, *Aspects of symmetries*, Cambridge University Press (1985); S. Treiman, R. Jackiw, B. Zumino, E. Witten *Current algebra and anomalies*, World Scientific, Singapore (1985), and to learn about integrable systems, R. Baxter *Exactly solved models in statistical mechanics*, Academic Press, New York (1982), and M. Gaudin *La fonction d'onde de Bethe*, Masson, Paris (1983). Of course, many more books are mentioned in the notes. We are also aware that several important texts are either in preparation or will appear in the near future.

Our knowledge of English remains to this day very primitive and we apologize for our cumbersome use of a foreign language. This lack of fluency has prevented us of any attempt at humour which would have been sometimes more than welcome.

We would have never undertaken writing, were it not for the teaching opportunities that we were given by several universities and schools. One of the authors (C.I.) is grateful to his colleagues from the "Troisième cycle de Suisse Romande" in Lausanne and Geneva, from the "Département de Physique de l'Université de Louvain La Neuve" and from the "Troisième cycle de physique théorique" in Marseille for giving him the possibility to teach what became parts of this text, as well as to the staff of these institutions for providing secretarial help in preparing a French unpublished manuscript. The other author (J.M.D.) acknowledges similar opportunities afforded by the "Troisième cycle de physique théorique" in Paris.

The final and certainly most pleasant duty is, of course, to thank all those, friends, colleagues, collaborators, students and secretaries who have helped us through the years. A complete list should include all the members of the Saclay "Service de physique théorique", together with its numerous visitors and the members of the many departments, institutions and meetings which offered us generous hospitality and stimulation.

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Saclay, 1988

Contents

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FROM BROWNIAN MOTION TO EUCLIDEAN FIELDS

It may seem surprising to start our study with a description of Brownian motion. However, this offers an interesting introduction to the concept of Euclidean quantum field, and an intuitive understanding of the role of dimensionality. The effective (or Hausdorff) dimension two of Brownian curves is particularly significant. It means that two such curves fail to intersect, hence to interact, in dimension higher than four. This is illustrated in the first section of this chapter, which also discusses the transition from a discrete to a continuous walk. A similar analysis for interacting fields, pioneered by K. Symanzik, is presented in the second section. It is related to rong coupling, or high temperature, expansions, to be studied later, in particular in chapter 6 of this volume and chapter 7 of volume 2. The introduction of n-component fields provides the means to incorporate "self-avoiding" walks in the limit $n \to 0$. We conclude this chapter with an analysis of elementary one-dimensional systems. This enables us to introduce the useful concept of transfer matrix.

1.1 Brownian motion

1.1.1 Random walks

We begin with a discussion of random walks on a regular, infinite lattice in d-dimensional Euclidean space. Each site has q neighbours, where q is called the *coordination number* of the lattice. At regular time intervals, separated by an amount $\Delta t = 1$, a walker jumps from one site towards a neighbouring one, chosen at random. The probability of landing on any particular adjacent site is $1/q$. Successive jumps are considered to be statistically independent events. We choose a (hyper)-cubic lattice generated

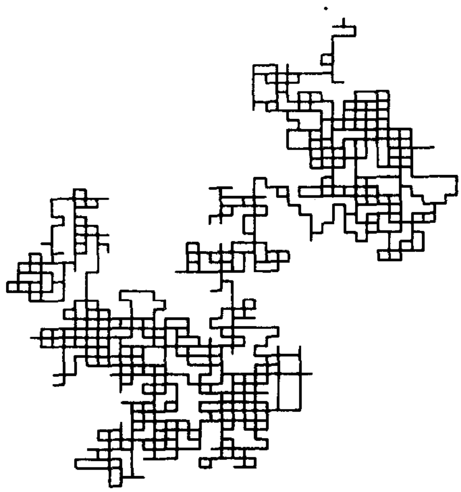

Fig. 1 A two-dimensional random walk of 2000 steps.

by d orthonormal vectors $\mathbf{e}_{(1)}$, ..., $\mathbf{e}_{(d)}$ such that $\mathbf{e}_{(\mu)} \cdot \mathbf{e}_{(\nu)} = \delta_{\mu\nu}$. Sites are located at $\mathbf{x} = x^\mu \mathbf{e}_{(\mu)}$, where the coordinates x^μ are integers. For this lattice, $q = 2d$. Figure 1 shows such a two-dimensional random walk of 2000 steps. Note the presence of large patches where the walker visited all sites.

Let us determine the conditional probability $P(\mathbf{x}_1, t_1; \mathbf{x}_0, t_0)$ for the walker to be on the site $\mathbf{x}_1$ at time t_1, knowing that his initial position was $\mathbf{x}_0$ at time t_0. For $t_1 = t_0$, we have

$$P(\mathbf{x}_1, t_0; \mathbf{x}_0, t_0) = \delta_{\mathbf{x}_1, \mathbf{x}_0} \tag{1}$$

The notation $\delta_{\mathbf{x}_1, \mathbf{x}_0}$ is shorthand for the product $\prod_{\mu=1}^{d} \delta_{x_1^\mu x_2^\mu}$. The probability P is defined for $t_1 \geq t_0$ and depends only on the differences $t_1 - t_0$, $\mathbf{x}_1 - \mathbf{x}_0$, because of the translational invariance in both time and space. At fixed t_1, the normalization condition reads

$$\sum_{\mathbf{x}_1} P(\mathbf{x}_1, t_1; \mathbf{x}_0, t_0) = 1 \tag{2}$$

and each probability P is either positive or zero.

The discrete formulation depends strongly on the choice of lattice. For instance, if $t_1 - t_0$ is even (odd), then so is the sum of coordinates of $\mathbf{x}_1 - \mathbf{x}_0$. However, we are only interested in those

$t=0$	$t=1$	$t=2$	$t=3$
1	1 1 · 1 1	1 2 · 2 1 · 4 · 1 2 · 2 1	1 3 · 3 3 · 9 · 3 1 · 9 · 9 · 1 3 · 9 · 3 3 · 3 1
1	4	16	64

Fig. 2 Relative probabilities for a two-dimensional random walk on a square lattice. The normalization factor is 4^{-t}.

asymptotic properties which are independent of the particular lattice structure chosen.

A recurrence relation between the probabilities P at successive times t and $t+1$ follows from the fact that the walker can reach the point $\mathbf{x}$ only if, one unit of time before, he was on a neighbouring site $\mathbf{x}' = \mathbf{x} \pm \mathbf{e}_{(\mu)}$

$$P(\mathbf{x}, t+1; \mathbf{x}_0, t_0) = \frac{1}{2d} \sum_{\substack{\mathbf{x}' \\ \text{neighbour of } x}} P(\mathbf{x}', t; \mathbf{x}_0, t_0) \qquad (3)$$

This relation generalizes Pascal's construction for the binomial coefficients, which corresponds to $d = 1$. Now P is completely determined by equations (1) to (3). Figure 2 illustrates the two-dimensional case.

Equation (3) can be rewritten, using a discretized version, Δ_r, of the Laplacian operator Δ,

$$\Delta_r f(\mathbf{x}) = \frac{1}{2d} \sum_{\mu=1}^{d} [f(\mathbf{x} + \mathbf{e}_{(\mu)}) + f(\mathbf{x} - \mathbf{e}_{(\mu)}) - 2f(\mathbf{x})] \qquad (4)$$

as

$$P(\mathbf{x}, t+1; \mathbf{x}_0, t_0) - P(\mathbf{x}, t; \mathbf{x}_0, t_0) = \Delta_r P(\mathbf{x}, t; \mathbf{x}_0, t_0) \qquad (5)$$

This is a finite difference approximation to the diffusion equation in continuous space

$$\left(\frac{\partial}{\partial t} - \Delta\right) P = 0 \qquad (6)$$

A Fourier transform allows one to solve equations (3) or (5). Writing

$$P(\mathbf{x},t;\mathbf{x}_0,t_0)=\int_{-\pi}^{\pi}\frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\tilde{P}(\mathbf{k},t) \tag{7}$$

we deduce

$$\tilde{P}(\mathbf{k},t+1)=\frac{1}{d}\sum_{\mu=1}^{d}\cos k_\mu \tilde{P}(\mathbf{k},t),\qquad \text{with}\quad \tilde{P}(\mathbf{k},t_0)=\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}_0}$$

as follows from the initial condition at $t=t_0$. Hence the solution is

$$P(\mathbf{x}_1,t_1;\mathbf{x}_0,t_0)=\int_{-\pi}^{\pi}\frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_0)}\left(\frac{1}{d}\sum_{\mu}\cos k_\mu\right)^{t_1-t_0} \tag{8}$$

We want to obtain the asymptotic properties of this solution at large distance with respect to the lattice spacing and for long times. It is convenient to rescale distances and time using a lattice spacing a rather than 1, and a time interval τ rather than 1. Now $\mathbf{e}_{(\mu)}\cdot\mathbf{e}_{(\nu)}=a^2\delta_{\mu\nu}$. Performing the substitutions $t\to t/\tau$, $\mathbf{x}\to\mathbf{x}/a$ and $\mathbf{k}\to a\mathbf{k}$, equation (8) becomes

$$P(\mathbf{x}_1-\mathbf{x}_0,t_1-t_0)=a^d\int_{-\pi/a}^{\pi/a}\frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_0)}\left(\frac{1}{d}\sum_{\mu}\cos ak_\mu\right)^{(t_1-t_0)/\tau} \tag{9}$$

We now take the limit as a and τ go to zero, keeping the distances and time intervals fixed. We consider a volume Δx around $\mathbf{x}$ which is large with respect to the elementary lattice volume a^d, but which is also sufficiently small to ensure that P remains nearly constant within Δx; this last requirement is fulfilled if $(t_1-t_0)/\tau$ is also large. This permits a probability density $p=P/a^d$ to be defined as

$$\begin{aligned}p(\mathbf{x}_1-\mathbf{x}_0,t_1-t_0)\Delta x&=\sum_{\mathbf{x}'_1\in\mathbf{x}_1+\Delta x}P(\mathbf{x}'_1-\mathbf{x}_0;t_1-t_0)\\&\approx\frac{\Delta x}{a^d}P(\mathbf{x}_1-\mathbf{x}_0,t_1-t_0)\end{aligned} \tag{10}$$