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Theory and Applications

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FOREWORD

This volume consists of 101 papers in the general areas of approximation, optimization and computing, and applications. Under the sponsorship of IMACS, it represents a collaborative venture, initiated in 1986, between the Dalian University of Technology and the University of Regina. A primary goal of the joint program and publication committee was to provide an opportunity for research papers reflecting emerging directions within theory or applications. The papers accepted for publication represent the varied and substantial efforts of 157 authors towards this goal.

The four invited papers are followed by 57 contributions in approximation theory. The two groupings for optimization (22) and computing (8) precede the collection of applications in several areas. Some papers could belong to more than one of the groupings and the final arrangement reflects the editors' decision.

A. G. Law and C.L. Wang
December, 1989

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MULTIPLE NUMERICAL INTEGRATION FORMULA USING REGULAR LATTICES

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Abstract

As is known, the trapezoidal rule with equidistant nodes has the highest accuracy when the influences at the end points may be neglected. Under the similar situations, we propose multiple numerical integration formulas over the vertices of regular dense lattices, such as equilateral triangular lattice on the plane, face-central cubic or body-central cubic lattices in R^3, or the D_4-lattice in R^4. We also give a method of error analysis and some numerical examples.

1. INTRODUCTION

Prof. M. Mori [7] proved that the trapezoidal rule with equidistant nodes has the highest accuracy among all the numerical quadrature formulas, when the influences at the end points may be neglected. Such situation occurs when the integrand decays very rapidly or the integration of a periodic function along a whole period. We would like to generalize it for multiple integrals.

In the Euclidean space R^n of higher dimension, the nodes of the simple cubic lattice SC_n are quite sparse. In fact, the centre of a hypercubic cell is apart $\sqrt{n}/2$ times to its side-length from the surrounding vertices, which is longer than the side when $n>5$. Therefore, we must select more dense regular lattices for multiple integration. Although the precise densest lattices in R^n are seldomly known, we have several dense lattices not difficult to construct. For example, we have the *equilateral triangular lattice* L_2 in R^2, the *face-central cubic lattice* FC_3 or the *body-central cubic lattice* BC_3 in R^3, and in R^4 a very dense lattice called *D_4-lattice* consisting of the vertices of tesselation by the regular 24 cells (e.g. Coxeter [1]). Also there are *E_8-lattice* in R^8 and the *Leech lattice* in R^{24}, though they are less useful for practical applications.

In the present paper, we first give our general formulas (§2) and discuss the construction of such regular lattices in normalized forms (§3). We then give a method of error analysis (§4). We shall mainly concern with the L_2-lattice on the plane, but the similar procedure holds for other lattices in higher dimensional spaces at least approximately. Finally in §5, we show some numerical examples.

The idea of using regular dense lattices for a multiple integral is not new. Such methods has been suggested in various literatures, such as [2], [3] and [5]. The author is particularly grateful to Prof. M. Sugihara whose lecture ([8]) at a meeting in RIMS has suggested the present algorithm. This paper is essentially a revised version of the note [4].

2. GENERAL FORMULAS

Let $\Omega = \Omega(L)$ be the set of vertices in a given lattice L in R^n. The *fundamental region* A_P of a vertex P is the set of points X in R^n such that the distance $\overline{PX}$ does not exceed the distance $\overline{QX}$ for any other vertices $Q \neq P$ in Ω. We say that the lattice L is *regular* if the volume of the fundamental region $V(A_P)$ does not depend upon the selection of the vertex P. We denote the constant volume by V.

Now, in order to compute approximately a multiple integral $\int_D f(x)dx$, we first cover the integration domain D by a regular lattice L,

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and take the summation over all lattice points x in D; say

(1) $$T = V \cdot \sum_{x \in D} f(x).$$

We mainly concern with a rapidly decreasing function $f(x)$, where the integral near the boundary ∂D may be almost negligible. If necessary, we can modify the integral near the boundary by the method in §4. Hence, we may assume that the integration domain D itself is a union of the fundamental regions A_P, $P \in V(L)$.

Our formula is actually a generalization of the *midpoint rule* to higher dimensional spaces; i.e., the integral over a fundamental region A_P is replaced by the product of the representative value of the integrand at its centre P and the volume V of A_P.

As we mainly concern with rapidly decreasing functions, it may be more convenient to arrange the set of vertices along the spherical layers around the origin. We denote by N the set of the square of the normilized distance of a vertex from the origin, except 0. For abbreviation, we shall call N simply the *set of distances*, although the actual distance is $\sqrt{c}$, $c \in N$.

Then our formula (1) reads

(2) $$T = V\,[f(0) + \sum_{c \in N} (\sum_{\|x\|^2 = c} f(x))].$$

For very regular lattices such as L_2, D_4 or E_8, the number of the vertices over a spherical layer $\|x\|^2 = c \neq 0$ is always a multiple of the number m of the nearest neighbouring verices in the lattice. This property does not hold for BC_3 and FC_3.

3. CONSTRUCTION OF THE REGULAR LATTICES

3.1. Equilateral triangular lattice on a plane

On the Euclidean plane R^2, we construct the equilateral triangular lattice L_2 as follows. We take one of the vertices to be the origin of the coordinates, and select one side issuing from it to be the x-axis. We denote by h the side-length of each equilateral triangle. Then each lattice point x of L_2 is uniquely represented by

$$X = (ae+bw)h$$

where e is the unit vector to the positive x-axis, $w = (1/2, \sqrt{3}/2)$, and a,b are integers. The number of the neighboring points is evidently 6. The fundamental region A is a regular hexagon of the side-length $h/\sqrt{3}$ with the centre at the lattice point. The volume V is $\sqrt{3}h^2/2$.

The set of distances N consists of the integers

$$c = a^2 + ab + b^2.$$

It is well-known that a number $c \in N$ as above is characterized by the properties that it is of the form $c = l^2 \cdot q$, where q is either 1 or square-free whose prime factors are 3 or primes $p \equiv 1 \pmod 6$. The first part of N is 1,3,4,7,9,12,13,16,19,21,25,27,28,31,36,37,39, 43,48,49,52,57,...

Further it is known that the primitive solutions of integers for $c^2=a^2+ab+b^2$ is given just by

$$a = (2m+n)n,\ b = m^2-n^2,\ c = m^2+mn+n^2$$

where m and n are relatively prime positive integers $(m>n)$ whose difference is not divisible by 3.

Assume that the integrand $f(x,y)$ is represented in the form

$$\sum_{i=1}^{m} \varphi_i(r) \cdot g_i(\theta)$$

in polar coordinates (r,θ), where $\varphi_i(r)$ is rapidly decreasing with respect to r, and $g_i(\theta)$ varies rather slowly in θ. Then the following summation procedure may be useful.

(3) $$T = \frac{\sqrt{3}}{2} h^2\{f(0,0) + \sum_{i=1}^{m} (\sum_{c \in N} \varphi_i(\sqrt{c}) \times \sum_{k=0}^{5} [g_i(\frac{k\pi}{3} + \theta_0) + g_i(\frac{k\pi}{3} - \theta_0)])\},$$

where $c = a^2+ab+b^2 \in N$; a,b are integers satisfying $0 \leq b \leq a$, $(a,b) \neq (0,0)$, and $\theta_0 = \arccos[(2a+b)/2r]$. The last term in (3) should be replaced by

$$\sum_{k=0}^{5} g_i(\frac{k\pi}{3} + \theta_0)$$

when $b = 0$ $(\theta_0 = 0)$ or $a = b$ $(\theta_0 = \pi/6)$. Note that the pair (a,b) is not unique for a given c, and the summation must be computed with respect to all indices (a,b).

3.2. Face-central and Body-central lattices in R^3

The vertices of a face-central cubic lattice FC_3 in R^3 are given, after suitable