A Mathematics Sampler

Topics for the Liberal Arts

THIRD EDITION

William P. Berlinghoff
& Kerry E. Grant

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To Phyllis and Sandy

PREFACE

This book grew out of a liberal-arts course at Southern Connecticut State University taken by nonscience majors, usually in their first year. Our experiments with the form and content of this course for more than a decade seemed to be most successful when several different mathematical areas were introduced to the students and covered in some depth. From these experiments emerged text units, which ultimately became chapters in this book. Thus, *A Mathematics Sampler* is a collection of independent chapters covering a broad spectrum of mathematics, old and new, pure and applied, traditional and unusual. The Second Edition of this book was adopted by more than 40 colleges and universities in its first two years in print. The course on which it is based was cited as an innovative approach to liberal-arts mathematics in Lynne Cheney's 1989 report, *50 HOURS: A Core Curriculum for College Students*, published by the National Endowment for the Humanities.

New Features

As a result of our teaching experiences and those of our colleagues around the country who have been kind enough to share their thoughts with us, we have revised the content of the text as follows:

- A chapter on graph theory (Chapter 9) has been added.
- The treatment of statistics in Chapter 4 has been thoroughly reworked and expanded.
- An optional section has been added to Chapter 8; it discusses the 4-space generalizations of cylinders and cones.

- Appendix B on the history of mathematics has been reworked to acknowledge the contributions of outstanding woman mathematicians and to reflect the emerging trends of the latter part of the 20th century.

Writing Exercises

In our view, the most important new feature of this edition — a feature that sets *A Mathematics Sampler* apart from all other books for this audience — is *the writing exercises that are spread throughout the book*! In recent years there has been widespread national interest in incorporating writing as a tool for learning science and mathematics. Writing assignments have always been an important part of the course as we have taught it. In this edition of the text we have included numerous writing exercises, many of them already class-tested by us. These exercises come in two forms: **(1)** At the end of virtually every section there is a short list of questions under the heading of **Writing Exercises**, most of which call for answers of more than a sentence and less than a page. These are suitable for use as homework or as in-class exercises to help focus on the ideas at hand, or even for incorporation into course journals. **(2)** At the end of each chapter there is a section entitled **Topics for Papers**. These are descriptions of longer assignments for more formal papers of 4–6 pages in length, or perhaps even more at times. They are intended to encourage the student to draw together the main ideas of the chapter and/or to extend those ideas to some independent exploration.

Necessary Preparation

The wide variety of student backgrounds in a first-year course for non-majors makes it unwise to require significant skill or sophistication as a prerequisite. We assume only that the student has had a first course in high-school algebra and is familiar with some elementary geometry and perhaps (though these are reviewed) some set-theoretic ideas. This book will not remedy a student's lack of basic skills. On the other hand, it *will* reinforce those skills by drawing on them in solving interesting problems relevant to the understanding of real mathematics and a real world.

LINK Sections

A primary function of an introductory mathematics course in a student's liberal education is to exhibit the interrelationships between mathematics and the various other fields of human knowledge. Too often we mathematicians tend to think of those interrelationships merely as applications of mathematical methods and results to other subject areas. While this is one kind of link between mathematics and other subjects, there are many others — links of analogy, of common historical influence, of personal achievement. These links provide a sense of the underlying unity of all knowledge and shed some light on both the power and the limitations of mathematics as a tool for understanding the world of our common experience and as an area of individual aesthetic achievement. *A Mathematics Sampler* provides LINK sections at the end of each topic, connecting specific mathematical ideas with many other subject areas. The connections examined are as varied as the fields explored — art, chemistry, coding, demographics, fiction, genetics, management, marketing, music, philosophy, politics, psychology, and social planning. These LINKs are intended to kindle a spark of interest, rather than to teach a technique, so they are brief, informal, and open-ended. They may be used for outside reading, as a basis for class discussions, or as leads for independent student projects.

Synopses of the Chapters

The book begins with an introduction to problem solving in mathematics, followed by an exploration of eight different mathematical topics. Each chapter after the first starts "from scratch" and proceeds to develop a significant mathematical idea, illustrating what mathematicians do in that area. *These eight chapters may be covered in any combination and in any order.* Some chapters have several natural stopping places to allow for either introductory coverage or in-depth exploration of the topic. Although the book contains enough material for a full-year (6-credit) course, it is most frequently used as a one-semester text, thereby affording considerable flexibility in the choice of topics to be covered. Three or four chapters, used in conjunction with Chapter 1 and the history appendix, provide ample material for a 3-credit course. In particular:

- **Chapter 1**, on problem-solving, sets the tone for the entire book. We recommend it as an appropriate first chapter in any choice of topics.
- **Chapter 2** investigates the topic of perfect numbers as a way of exhibiting the power and elegance of number theory. The students see that analytical observation, educated guesswork, and trial-and-error investigation are good sources of mathematical conjecture, but that patterns can be deceiving. To be most effective, this chapter should be covered in its entirety, although some of the proofs along the way are easily omitted.
- **Chapter 3** uses geometry as the vehicle for examining the axiomatic structure of formal mathematics. After a brief overview of the principles of Euclidean geometry, the historical controversy over Euclid's parallel postulate is used as a steppingstone to consistency and independence in formal axiom systems, culminating in a discussion of the modern view of the relationship between pure mathematical systems and real-world models.
- **Chapter 4** begins with a brief treatment of set language and notation, motivated by their utility in discussing probability questions. (Section 4.2 may be used independently as a review of or an introduction to basic set-theoretic ideas.) The next two sections present the fundamental idea of probability and some combinatoric principles. The chapter may be terminated after Section 4.5 if those ideas are

sufficient for the needs of the course. Sections 4.6 and 4.7 provide a further exploration of probability, and LINK Section 4.8 is another natural intermediate stopping place. Sections 4.9 through 4.12 examine some fundamental ideas of statistics, including normal distributions and the Central Limit Theorem.

- **Chapter 5** introduces students to microcomputers. Assuming no prior computer knowledge, this chapter develops enough BASIC programming so that students can actually construct and run some simple interactive "video games" by the end of Section 5.6. A cursory treatment of this topic might legitimately stop after Section 5.4 or 5.5, but we recommend including Section 5.7 in any event.
- **Chapter 6** begins with the elementary notions of *set* and *1-1 correspondence*, and progresses to an explanation and proof of Cantor's Theorem. Discussions of paradoxes, the Continuum Hypothesis, and philosophies of mathematics are provided. The proof of Cantor's Theorem in Section 6.7 may be omitted without seriously affecting the integrity of the chapter. A brief treatment might bypass the computational techniques of Section 6.4 and end early in Section 6.7, but unless done with care, it runs the risk of obscuring the entire point of Cantor's work.
- **Chapter 7** takes as its topical goal the explanation and proof of Lagrange's Theorem, exhibiting the power and utility of abstraction. Tables for finite groups are used to examine properties of operations. Section 7.8, the proof of Lagrange's Theorem, may be omitted without disturbing the continuity of the chapter. A brief treatment of this chapter could end with Section 7.5 or 7.6.
- **Chapter 8** introduces four-dimensional geometry by way of analogy with lower dimensions. The earlier sections provide practice and review of the concepts of *coordinate*, *interval*, *path*, and *distance* in 1-, 2-, and 3-space, which are generalized so that visual intuition may be replaced naturally by coordinate-algebraic techniques in making the transition to 4-space. Sections 8.6 and 8.7 extend these ideas to several different types of figures; one or both of these sections may be omitted, if desired.
- **Chapter 9** investigates the twin existence problems of an Euler path and a Hamilton circuit in a graph. The problems are easily understood and appear to be quite similar in form. The radical dissimilarity of results, by contrast, provides an instructive insight into the nature of mathematics. The LINK section illustrates how graph theory is applied to project management.

- **Appendix A** is a self-contained, concise reference unit on the basic principles of mathematical logic. It can be taught as a short unit in its own right, it can be used as an independent study assignment, or it can just be consulted as needed for definitions or examples during the course.
- **Appendix B** is a compact history of mathematics in relation to the major events that shaped the development of Western civilization, from the earliest known evidence of mathematical achievements through the latter part of the 20th century. The authors recommend it as an early outside-reading assignment. This material can then be used throughout the course as a historical context in which to place the ideas covered in the other chapters.

Answers and/or hints to most of the odd-numbered exercises are provided in the back of the book. Examples are numbered by section; figures and tables are numbered consecutively by chapter. We have adopted the typographical convention of signaling the end of each example with the symbol "□."

Gratitude

As preparation of this text progressed through various stages, many people provided help and encouragement. We are grateful to Ross Gingrich, Michael Meck, Dorothy Schrader, Michael Shea, J. Philip Smith, and other colleagues at Southern Connecticut State University for their expert advice, to Martin Zuckerman of City College of the City University of New York for a wealth of helpful suggestions, and to Joseph Moser and his colleagues at West Chester University for their comments on earlier editions of this book. Special thanks to H. T. (Pete) Hayslett, Jr., and Dexter Whittinghill of Colby College for their valuable assistance during the expansion and reworking of the statistics sections of Chapter 4, and to Claudia Henrion of Middlebury College for steering us to relevant material on women in mathematical history.

William P. Berlinghoff
Kerry E. Grant

TO THE STUDENT

As you begin this topical tour of mathematics, we who seek to guide you have a few words of advice and perspective to offer. The mathematics in this book is not highly technical, but the concepts often are challenging. Mastery of them will at times test your patience and perseverance. We hope you will find our writing style comfortable and our explanations clear. But do not expect to read this book — or any mathematics book — like a novel. Expect to read and reread thoughtfully. Build a habit of having paper and pencil at hand to answer questions raised in the text, to work through exercises, and to create examples of your own. Examine your own understanding frequently. Do not go on to new material until you understand the old, or at least until you know exactly what it is that you do not understand. And do not hesitate to ask questions of your instructor!

A word of warning: Just doing the assigned exercises is *not* your main job in this course! If you regard any text material that does not explain "how to" as superfluous, you have fallen into a dangerous trap. The primary purpose of this book is not to refresh old mathematical techniques or to teach you new ones, even though both of those things may well occur in the course. Rather, we seek to show you some mathematical ideas you may not have seen before and may not even regard as part of mathematics. The text discussions you will read and the exercises you will do are important details in these larger pictures, like pieces of a jigsaw puzzle that make little sense until you begin to see the outlines of the picture as a whole. As you work your way along, then, spend some time thinking about how these details fit into the larger picture of the topic you are studying. When you recognize and understand the broad view of the landscape, you will have arrived at your destination. We hope you enjoy the trip!

W. P. B.
K. E G.

CONTENTS

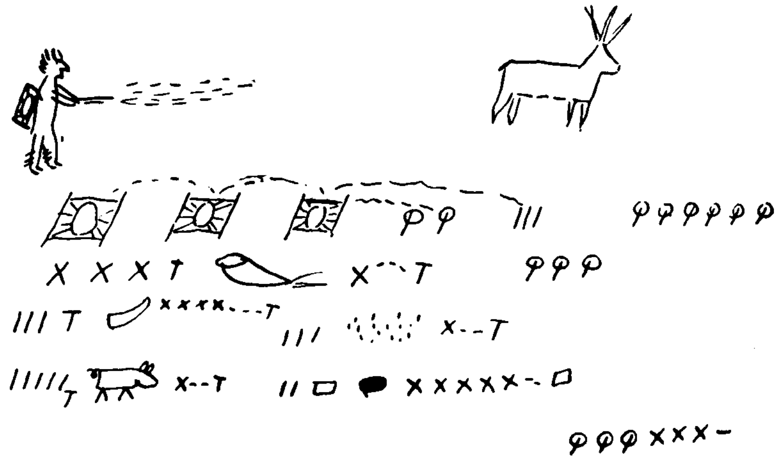
CHAPTER

1 PROBLEMS AND SOLUTIONS

1.1 What Is Mathematics?

This book is written by two mathematicians. Because we are mathematicians and because mathematicians are fond of precise definitions and careful logic, we would like to begin by defining "mathematics" for you. Then we could proceed to unfold that definition to exhibit in logical sequence the various topics covered in the rest of the book. We would like to do that—but we can't. Mathematics, a subject utterly dependent on definitions and logic, cannot itself be defined in a clear, comprehensive sentence or two, or even in a short paragraph. In fact, mathematicians themselves often delight in giving examples to show that any proposed definition of their subject is deficient in one way or another. The many diverse ideas, methods and results of modern mathematics defy simple description. Faced with this ironic situation, how, then, are we to make any sense at all of the label "mathematics"?

Let us look first at what mathematics has been. At different times and different places in the early history of mankind, civilization brought with it the need for counting and measuring. Traders needed to tally their goods and profits; builders needed to cut their stone and wood to exact sizes; landowners needed to mark precisely the edges and corners of their fields; navigators needed to fix their positions by using the stars to compute distances and angles; the list goes on and on. These specialized tasks gave rise to skilled craftsmen of various sorts—counting-house clerks, draftsmen, surveyors, astronomers, etc. Their crafts, though different, had some common features: They all used numbers and/or basic shapes, and they all sought precise quantitative descriptions of some part of their world. This array of



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Accounting by the Passamaquoddy Division of the Abnaki Tribe.

specialized counting and measuring techniques was all of mathematics for the first several thousand years of recorded history.

Greek philosophical speculation during the five or six centuries before Christ brought about a major change in mathematics. As the Greeks tried to solve the mysteries of truth, beauty, goodness, and life itself, they turned human reason into formal logic and then applied that logical system to their observations of the world. Their belief that logic was the key to understanding reality led them to apply it to all areas of thought, especially to mathematics. As they systematically recast the techniques of counting and measuring into logical systems based on "self-evident" assumptions about the real world, mathematics changed from craft to science. Complex mathematical statements were deduced from simpler ones, and they all were applied to solve the problems of the physical world. Architecture, surveying, astronomy, mechanics, optics—all these fields grew from mathematics applied to solve real-world problems, and those solutions provided observable confirmation that the physical world behaved as mathematics predicted it would. Thus, for more than two thousand years mathematics was regarded as the "queen of sciences," the only reliable means for finding scientific truth.

Then came the 19th century. The development of the non-Euclidean geometries in the early 1800s provided mankind with three separate geometric systems, each one logically correct by itself, each one providing a perfectly good way of describing physical space, and each one contradicting the other two! If mathematics were truly a science—that is, if mathematical truths were experimentally verifiable facts about the real world—then *this* paradoxical situation could never occur. Mathematics, therefore, must be a study that is essentially independent of the real world, and hence its pursuit


Ein ander Exempel.

Ein Fraw oder Haußmutter gehet auff den marckt/ kaufft vberhaupt ein Körblin mit Rebnerbyrn/darumb gibt sie achtzehen pfenning / so sie heim kompt / findet sie im körblin hundert vnd achtzig byrn / Ist die frag/wie vil byren sie vmb ein pfenning habe? Thu/als obgelert/ so kompt dir zehen/ Also vil byren hat sie vmb einen pfenning / Vnd ist wolfeyl drumb.

The Problem of the Market Woman. *From Köbel's Rechenbüchlein of 1514 (1564 edition). Traders needed to tally their goods and profits.*

need not be bound by human experience. Mathematicians suddenly found themselves free to explore any questions their minds could ask, unhampered by worries about whether their results had anything to do with reality. They found themselves in "a world of pure abstraction;... 'the wildness of logic' where reason is the handmaiden and not the master."[1] Mathematics became a subject essentially different from the (other) sciences: "In other sciences the essential problems are forced upon the subject from external sources, and the scientist has no control over the ultimate end. The mathematician, however, is free to prescribe not only the means of realizing the end, but also the end itself."[2]

Mathematics responded to this newfound freedom with a mushroomlike expansion in all directions. Results began to appear at such a rate that by the 1980s mathematicians were publishing more than 200,000 *new* theorems every year! Some of these theorems solve old problems, but most of them solve new ones, problems suggested by results published only a short time earlier or even suggested for the first time in that same paper. Some of these theorems are profoundly significant, some are utterly useless, most are somewhere in between—and no one really knows (yet) which is which. Modern mathematics has become an art, the art of posing and solving problems

[1] Marston Morse, "Mathematics and the Arts." *The Yale Review,* 40 (4): 1951, p. 612.

[2] Raymond G. Ayoub, in a book review of *Mathematics: The Loss of Certainty. MAA Monthly* 89 (9): 1982, p. 716.