FAST TRANSFORMS

Algorithms, Analyses, Applications

Douglas F. Elliott

K. Ramamohan Rao

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Douglas F. Elliott

Electronics Research Center
Rockwell International
Anaheim, California

K. Ramamohan Rao

Department of Electrical Engineering
The University of Texas at Arlington
Arlington, Texas

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PREFACE

Fast transforms are playing an increasingly important role in applied engineering practices. Not only do they provide spectral analysis in speech, sonar, radar, and vibration detection, but also they provide bandwidth reduction in video transmission and signal filtering. Fast transforms are used directly to filter signals in the frequency domain and indirectly to design digital filters for time domain processing. They are also used for convolution evaluation and signal decomposition. Perhaps the reader can anticipate other applications, and as time passes the list of applications will doubtlessly grow.

At the present time to the authors' knowledge there is no single book that discusses the many fast transforms and their uses. The purpose of this book is to provide a single source that covers fast transform algorithms, analyses, and applications. It is the result of collaboration by an author in the aerospace industry with another in the university community. The authors hope that the collaboration has resulted in a suitable mix of theoretical development and practical uses of fast transforms.

This book has grown from notes used by the authors to instruct fast transform classes. One class was sponsored by the Training Department of Rockwell International, and another was sponsored by the Department of Electrical Engineering of The University of Texas at Arlington. Some of the material was also used in a short course sponsored by the University of Southern California. The authors are indebted to their students for motivating the writing of this book and for suggestions to improve it.

The development in this book is at a level suitable for advanced undergraduate or beginning graduate students and for practicing engineers and scientists. It is assumed that the reader has a knowledge of linear system theory and the applied mathematics that is part of a standard undergraduate engineering curriculum. The emphasis in this book is on material not directly covered in other books at the time it was written. Thus readers will find practical approaches not covered elsewhere for the design and development of spectral analysis systems.

The long list of references at the end of the book attests to the volume of literature on fast transforms and related digital signal processing. Since it is impractical to cover all of the information available, the authors have tried to list as many relevant references as possible under some of the topics discussed only briefly. The authors hope this will serve as a guide to those seeking additional material.

Digital computer programs for evaluation of the transforms are not listed, as these are readily available in the literature. Problems have been used to convey information by means of the format: If A is true, use B to show C. This format gives useful information both in the premise and in the conclusion. The format also gives an approach to the solution of the problem.

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LIST OF ACRONYMS

ADC	Analog-to-digital converter
AGC	Automatic gain control
BCM	Block circulant matrix
BIFORE	Binary Fourier representation
BPF	Bandpass filter
BR	Bit reversal
BRO	Bit-reversed order
CBT	Complex BIFORE transform
CCP	Circular convolution property
CFNT	Complex Fermat number transform
CHT	Complex Haar transform
CMNT	Complex Mersenne number transform
CMPY	Complex multiplications
CNTT	Complex number theoretic transform
CPFNT	Complex pseudo-Fermat number transform
CPMNT	Complex pseudo-Mersenne number transform
CRT	Chinese remainder theorem
DAC	Digital-to-analog converter
DCT	Discrete cosine transform
DDT	Discrete D transform
DFT	Discrete Fourier transform
DIF	Decimation in frequency
DIT	Decimation in time
DM	Dyadic matrix
DST	Discrete sine transform
ENBR	Effective noise bandwidth ratio
ENBW	Equivalent noise bandwidth
EPE	Energy packing efficiency
FDCT	Fast discrete cosine transform
FFT	Fast Fourier transform
FGT	Fast generalized transform
FIR	Finite impulse response
FNT	Fermat number transform
FOM	Figure of merit
FWT	Fast Walsh transform
GCBC	Gray code to binary conversion
GCD	Greatest common divisor
GT	Generalized discrete transform
$(GT)_r$	rth-order generalized discrete transform
HHT	Hadamard–Haar transform
$(HHT)_r$	rth-order Hadamard–Haar transform
HT	Haar transform
IDCT	Inverse discrete cosine transform
IDFT	Inverse discrete Fourier transform
IF	Intermediate frequency
IFFT	Inverse fast Fourier transform
IFNT	Inverse Fermat number transform
IGT	Inverse generalized transform
$(IGT)_r$	rth-order inverse generalized transform
IIR	Infinite impulse response
IMNT	Inverse Mersenne number transform
KLT	Karhunen–Loêve transform
LPF	Low pass filter
lsb	Least significant bit
lsd	Least significant digit
MBT	Modified BIFORE transform
MCBT	Modified complex BIFORE transform
MGT	Modified generalized transform
$(MGT)_r$	rth-order modified generalized discrete transform
MIR	Mixed radix integer representation
MNT	Mersenne number transform
MPY	Multiplications
msb	Most significant bit
msd	Most significant digit
mse	Mean-square error

MWHT	Modified Walsh Hadamard transform
$(MWHT)_h$	Hadamard ordered modified Walsh Hadamard transform
NPSD	Noise power spectral density
NTT	Number theoretic transform
PSD	Power spectral density
RF	Radio frequency
RHT	Rationalized Haar transform
RHHT	Rationalized Hadamard-Haar transform
$(RHHT)_r$	rth-order rationalized Hadamard-Haar transform
RMFFT	Reduced multiplications fast Fourier transform
RMS	Root mean square
RNS	Residue number system
RT	Rapid transform
SHT	Slant-Haar transform
$(SHT)_r$	rth-order slant-Haar transform
SIR	Second integer representation
SNR	Signal-to-noise ratio
ST	Slant transform
WFTA	Winograd Fourier transform algorithm
WHT	Walsh Hadamard transform
$(WHT)_{cs}$	Cal-sal ordered Walsh-Hadamard transform
$(WHT)_h$	Hadamard ordered Walsh-Hadamard transform
$(WHT)_p$	Paley ordered Walsh-Hadamard transform
$(WHT)_w$	Walsh ordered Walsh-Hadamard transform
zps	Zero crossings per second

NOTATION

Symbol	Meaning
$A, B, \ldots$	Matrices are designated by capital letters
$A \otimes B$	The Kronecker product of A and B (see Appendix)
A^T	The transpose of matrix A
A^{-1}	The inverse of matrix A
$[A(L)]$	DCT matrix of size $(2^L \times 2^L)$
$D(f)$	Periodic DFT filter frequency response, which for $P = 1$ s is given by $\exp\left[-j\pi f\left(1 - \frac{1}{N}\right)\right]\frac{\sin(\pi f)}{N\sin(\pi f/N)}$
$\hat{D}(f)$	Periodic frequency response of DFT with weighted input (windowed output)
$D'(f)$	Nonperiodic DFT filter frequency response which for $P = 1$ s is given by $\exp[-j\pi f(1 - 1/N)][\sin(\pi f)]/(\pi f)$
$\hat{D}'(f)$	Nonperiodic frequency response of DFT with weighted input (windowed output)
DFT$[x(n)]$	The discrete Fourier transform of the sequence $\{x(0), x(1), \ldots, x(N-1)\}$
$[D_r^j(L)]$	jth matrix factor of $[G_r(L)]$
E	Expectation operator
$[E_r^j(L)]$	jth matrix factor of $[M_r(L)]$
F_t	tth Fermat number, $F_t = (2^{2^t} + 1)$, $t = 0, 1, 2, \ldots$
$[G_r(L)]$	$(\mathrm{GT})_r$ matrix of size $(2^L \times 2^L)$
$[H_{\mathrm{mh}}(L)]$	MWHT matrix of size $(2^L \times 2^L)$
$[H_s(L)]$	Walsh Hadamard matrix of size $(2^L \times 2^L)$. The subscript s can be w, h, p, or cs, denoting Walsh, Hadamard, Paley or cal sal ordering, respectively.
$[\mathrm{Ha}(L)]$	Haar matrix of size $(2^L \times 2^L)$
$[\mathrm{Hh}_r(L)]$	rth order $(\mathrm{HHT})_r$ matrix of size $(2^L \times 2^L)$
$\bar{I}_m$	Opposite diagonal matrix, e.g., $\bar{I}_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$
$\vec{I}_N^{cm}$	Columns of I_N are shifted circularly to the right by m places
$\overleftarrow{I}_N^{cm}$	Columns of I_N are shifted circularly to the left by m places
I_N^{dl}	Columns of I_N are shifted dyadically by l places
I_R	Identity matrix of size $(R \times R)$
Im[ ]	The imaginary part of the quantity in the square brackets
IDFT$[X(k)]$	The inverse discrete Fourier transform of the sequence $\{X(0), X(1), \ldots, X(N-1)\}$
$[K(L)]$	KLT matrix of size $(2^L \times 2^L)$
L	Integer such that $N = \alpha^L$
M_P	Mersenne number, $M_P = 2^P - 1$, where P is a prime number

Symbol	Meaning
$[M_r(L)]$	$(\mathrm{MGT})_r$ matrix of size $(2^L \times 2^L)$
N	Transform dimension
N^{-1}	Multiplicative inverse of the integer N such that $N \times N^{-1} \equiv 1$ (modulo M)
P	1. Period of periodic time function in seconds 2. In Chapter 11, prime number
$[P(L)]$	Diagonal matrix whose diagonal elements are negative integer powers of 2
$P_h(m)$	$(\mathrm{WHT})_h$ circular shift-invariant power spectral point
$P_r(l)$	lth power spectral point of $(\mathrm{GT})_r$
$P_w(m)$	mth sequency power spectrum
Q	1. Ratio of the filter center frequency and the filter bandwidth (Chapter 6) 2. Least significant bit value (Chapter 7)
Re[ ]	The real part of the quantity in the square brackets
$R(D)$	Rate distortion
$[\mathrm{Rh}(L)]$	RHT matrix of size $(2^L \times 2^L)$
$[S(L)]$	ST matrix of size $(2^L \times 2^L)$
$[\vec{S}^{(cm)}(L)]$	Shift matrix relating $\vec{\mathbf{X}}^{(cm)}$ and $\mathbf{X}$
$[\tilde{S}^{(cm)}(L)]$	Shift matrix relating $\tilde{\mathbf{X}}^{(cm)}$ and $\mathbf{X}$
$[S^{(dl)}(L)]$	Shift matrix relating $\mathbf{X}^{(dl)}$ and $\mathbf{X}$
$[\mathrm{Sh}_r(L)]$	rth order $(\mathrm{SHT})_r$ matrix of size $(2^L \times 2^L)$
T	Sampling interval
W	1. $\exp(-j2\pi/N)$ for FFT 2. $\exp(-j2\pi/\alpha^{r+1})$ for FGT
$W^{(\cdot)}$	The element $\cdot$ in a matrix means $-j\infty$ so that $W^{(\cdot)} = W^{-j\infty} = e^{-\infty} = 0$
W^{A+B}	Shorthand notation for matrix product $W^A W^B$, where A and B are $N \times N$ matrices
W^E	Matrix with entry $W^{E(k,n)}$ in row k and column n, where E is a matrix of size $(N \times N)$, $E(k,n)$ is the entry in row k and column n for $k, n = 0, 1, \ldots, N-1$
$X(f)$ or $X_a(f)$	Spectrum defined by the Fourier (or generalized) transform of the (analog) function $x(t)$
$\lvert X(f)\rvert^2$	Power spectral density with units of watts per hertz
$X(k)$	Coefficient number k, $k = 0, \pm 1, \pm 2, \ldots$, in series expansion of periodic function $x(t)$
$\lvert X(k)\rvert^2$	Power spectrum for a function with a series representation
$\mathbf{X}_c$	DCT of $\mathbf{x}$
$\mathbf{X}_{cf}$	CFNT of $\mathbf{x}$
$\vec{\mathbf{X}}^{(cm)}$	Transform of $\vec{\mathbf{x}}^{cm}$
$\tilde{\mathbf{X}}^{(cm)}$	Transform of $\tilde{\mathbf{x}}^{cm}$
$\mathbf{X}_{cm}$	CMNT of $\mathbf{x}$
$\mathbf{X}_{cpf}$	CPFNT of $\mathbf{x}$
$\mathbf{X}_{cpm}$	CPMNT of $\mathbf{x}$
$\mathbf{X}^{(dl)}$	Transform of $\mathbf{x}^{dl}$
$\mathbf{X}_f$	FNT of $\mathbf{x}$
$\mathbf{X}_{ha}$	HT of $\mathbf{x}$
$\mathbf{X}_{hhr}$	$(\mathrm{HHT})_r$ of $\mathbf{x}$
$\mathbf{X}_k$	KLT of $\mathbf{x}$
$\mathbf{X}_m$	MNT of $\mathbf{x}$
$\mathbf{X}_{mh}$	MWHT of $\mathbf{x}$
$\mathbf{X}_{mr}$	$(\mathrm{MGT})_r$ of $\mathbf{x}$
$\mathbf{X}_{pf}$	PFNT of $\mathbf{x}$
$\mathbf{X}_{pm}$	PMNT of $\mathbf{x}$
$\mathbf{X}_r$	$(\mathrm{GT})_r$ of $\mathbf{x}$
$\tilde{\mathbf{X}}_r^{(cm)}$	$(\mathrm{GT})_r$ of $\tilde{\mathbf{x}}^{cm}$
$\mathbf{X}_{rh}$	RHT of $\mathbf{x}$
$\mathbf{X}_s$	ST of $\mathbf{x}$
$X_s(k)$	kth WHT coefficient. The subscript s is defined in $[H_s(L)]$
$\mathbf{X}_{shr}$	$(\mathrm{SHT})_r$ of $\mathbf{x}$
Z_M	Ring of integers modulo M represented by the set $\{0, 1, 2, \ldots, M-1\}$
Z_M^c	Ring of complex integers. If $c = a + jb$, where $a = \mathrm{Re}[c]$ and $b = \mathrm{Im}[c]$, then c is represented in Z_M^c by $\hat{a} + j\hat{b}$, where $\hat{a} = a \bmod M$ and $\hat{b} = b \bmod M$
$a \leftarrow b$	Give variable a the value of expression b (or replace a by b)
$a \in B$	a is an element of the set B
$a \in [c, d)$	$c \leqslant a < d$
comb_T	The infinite series of impulse functions defined by

Symbol	*Meaning*
	$\sum_{k=-\infty}^{\infty} \delta(t-kT)$
$\text{cube}[t/p]$	Cubic-shaped function defined by $\text{cube}\left[\frac{t}{p}\right] = \text{tri}\left[\frac{t}{P/2}\right] * \text{tri}\left[\frac{t}{P/2}\right]$
$\deg[\ ]$	The degree of the polynomial in the square brackets
f	Frequency in hertz
f_k	Digit in expansion of $f = \sum_{k=l}^{m} f_k \alpha^k$, where l is the least significant digit (lsd) and m is the most significant digit (msd)
f_s	$f_s = 1/T$ is the sampling frequency
$\langle ft \rangle$	$\sum_k q_k t_{-k}$
$h_s(k, n)$	Element of $[H_s(L)]$ in row k and column n. The subscript s is defined in $[H_s(L)]$
j	$\sqrt{-1}$
k	Transform coefficient number
$\langle\langle k \rangle\rangle$	The decimal number obtained by the bit reversal of the L bit binary representation of k
$\bar{k} \cdot \bar{s}$	The integer defined by $\sum_{l=0}^{r+1} k_{r+1-l} 2^{s-l}$, where $s = r+2, r+3, \ldots, L$, $k = 2^r, 2^{r+1}, \ldots, (2^{r+1}-1)$, and $k_l, l = 0, 1, \ldots, r+1$, is a bit in the binary representation of k
ln	Logarithm to the base e (natural logarithm)
log	Logarithm to the base 10
$\log_2$	Logarithm to the base 2
n	Data sequence number
q_k	Integerization of frequency given by $q_k = \left\| \sum_{l=-\infty}^{\infty} f_{k-r-1+l} a_l \right\|$

Symbol	*Meaning*
r	Integer in the set $(0, 1, 2, \ldots, L-1)$
$\text{rad}(m, t)$	mth Rademacher function
$\text{rect}[t/P]$	Rectangular-shaped function defined by $\text{rect}\left[\frac{t}{P}\right] = \begin{cases} 1, & \lvert t \rvert \leqslant P/2 \\ 0, & \text{otherwise} \end{cases}$
$\text{rep}_{f_s}[X(f)]$	The repetition of $X(f)$ every f_s units as defined by the convolution $X(f) * \text{comb}_{f_s}$
s	Seconds
$\text{sinc}(fQ)$	$[\sin(\pi f Q)]/(\pi f Q)$
t	Time in seconds
$\text{tr}[\ ]$	Trace of a matrix
$\text{tri}[t/P]$	Triangular-shaped function defined by $\text{tri}\left[\frac{t}{P}\right] = \text{rect}\left[\frac{t}{P/2}\right] * \text{rect}\left[\frac{t}{P/2}\right]$
$u(t-t_0)$	Unit step function defined by $u(t-t_0) = \begin{cases} 1, & t \geqslant t_0 \\ 0, & \text{otherwise} \end{cases}$
$\text{wal}_s(k, t)$	kth Walsh function. The subscript s is defined in $[H_s(L)]$
x^*	Complex conjugate of x
$\overleftarrow{\mathbf{x}}^{cm}$	$\mathbf{x}$ shifted circularly to the left by m places
$\overrightarrow{\mathbf{x}}^{cm}$	$\mathbf{x}$ shifted circularly to the right by m places
$\mathbf{x}^{dl}$	$\mathbf{x}$ is shifted dyadically by l places
$x(n)$	Sampled-data value of x for sample number n
$x(n) \leftrightarrow X(k)$	Both $x(n)$ and $X(k)$ exist
$x(t)$	Time domain scalar-valued function at time t
$\mathbf{x}(t)$	Time domain vector-valued function at time t
$x(t) \leftrightarrow X(f)$	Both $x(t)$ and $X(f)$ exist
$x_s(t)$	Sampled function
$x * y$	The convolution of x and y
$\mathbf{x} \circ \mathbf{y}$	Element by element multiplication of the elements in $\mathbf{x}$ and $\mathbf{y}$, e.g., if $\mathbf{a} = \mathbf{x} \circ \mathbf{y}$, then $a(k) = x(k)y(k)$
$(x)_\alpha$	Expression for x in number system with radix α, e.g., $(10.1)_2 = (2.5)_{10}$

Symbol	*Meaning*
z	$\mathscr{F}[\delta(t-T)] = \exp(-j2\pi fT)$
$\mathscr{F}$	Fourier transform operator
$\mathscr{R}(a/b)$	The remainder when a is divided by b
$\mathscr{G}$	Generalized transform operator
$\mathscr{W}(f)$	Fourier transform of $w(t)$
$a, b, \ldots$	Script lower case letters a, b, ... and the italic letters i, k, l, m, n, p, q, r (Chapter 5 only), K, L, M, and N denote integers
$a \equiv b$ (modulo n)	$\mathscr{R}(a/n) = \mathscr{R}(b/n)$, where a and b are either integers or polynomials
a mod b	$\mathscr{R}(a/b)$, where a and b are either integers or polynomials
$l \mid N$	l divides N, i.e., the ratio N/l is an integer and the set of such integers includes 1 and N
s	Steps per second taken by the generalized transform basis functions
$w(t)$	Weighting function applied to modify DFT filter frequency response
$[\Sigma_x(L)]$	Covariance matrix of **x**
α	Number system radix or a primitive root of order N
α^{r+1}	Number of equal sectors on the unit circle in the complex plane with first sector starting on the positive real axis
δ_{kl}	Kronecker delta function with the property that $\delta_{kl} = \begin{cases} 0, & k \neq l \\ 1, & k = l \end{cases}$
$\delta(t - t_0)$	Dirac delta function with the property that $x(t_0) = \int_{-\infty}^{\infty} \delta(t - t_0)x(t)\,dt$
$\theta_r(l)$	lth phase spectral point of $(\mathrm{GT})_r$
λ_j	jth eigenvalue of $[\Sigma_x(L)]$
μ	$E[x]$
ρ	Correlation coefficient
σ^2	$E[(x-\mu)^2]$
$\phi(N)$	The number of integers less than N and relatively prime to N
$\phi_k(n)$	kth basis function $\phi_k(t)$ evaluated at $t = nT$
$\lvert(\cdot)\rvert$	Magnitude of $(\cdot)$
$\lVert(\cdot)\rVert$	Integerize by truncation (or rounding)
$\lceil(\cdot)\rceil$	Smallest integer $\geqslant (\cdot)$, e.g., $\lceil 3.5 \rceil = 4$, $\lceil -2.5 \rceil = -2$
$\lfloor(\cdot)\rfloor$	Largest integer $\leqslant (\cdot)$, e.g., $\lfloor 3.5 \rfloor = 3$, $\lfloor -2.5 \rfloor = -3$
$\oplus$	Signed digit addition performed digit by digit modulo α

CONTENTS

Chapter 3 **Discrete Fourier Transforms**

Chapter 4 **Fast Fourier Transform Algorithms**

Chapter 5 **FFT Algorithms That Reduce Multiplications**