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METHODS OF OPTIMIZATION

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PREFACE

Optimization problems are encountered in many spheres of human life. Every rational action is optimal in some sense, because it is, as a rule, chosen after one has compared it with other possible variants. The interest in problems of optimal choice has always been high, but it has grown enormously in recent years due to widespread industrial application. On the one hand, there is increasing pressure to seek the most effective utilization of available resources. On the other hand, advances in computer technology have increased the scope of computer control of processes. Because of the complexity of problems in applications decision-making is of necessity based less and less on intuition or experience. Instead, the scientific approach drawing on conscious mathematical models is becoming unavoidable.

Problems involving extremal properties of geometric figures (circle, square, etc.) were studied and resolved already in antiquity. A powerful impetus to the development of optimization methods was the creation of differential calculus. In the eighteenth century the calculus of variations was developed. As a result of the industrial needs, the theory as well as practice of optimization began to develop rapidly. In the course of a short period of time, new areas of the theory emerged (linear programming, optimal control theory, among others), which have led to effective numerical methods of solving various extremal problems arising in practice. The study of optimization methods is evolving continuously and the sphere of their application is constantly widening.

In our book we present the basic methods used currently to solve various optimization problems. The emphasis is on the major problem, with detailed description of established, classical methods. Some techniques are discussed and illustrated by concrete problems demonstrating general principles.

The second edition of this textbook is different from the first in content, form, as well as order of presentation. We have drawn on our experience in teaching the material at the Department of Applied Mathematics of Belorussian State University, and incorporated results of recent research.

We begin by describing the classical methods of linear programming. In the first edition this material followed the general results of nonlinear and convex programming. In the second edition it is the core of the course. This order of presentation has allowed us to combine optimization with linear programming, taught separately in some universities. The results of Chapter 1 are extensively and repeatedly used in other chapters in studying general problems of optimization. We have developed the compact and elementary treatment, which does not require the theory of inequalities or the theory of convex polygons. As a result, the structure of the books has changed. The simplex method is given first in the form convenient for computer implementation, and the traditional tabular form, which has not been used in computer codes for some time, is explained later in solving examples. The duality theory of linear programming (Section 1.2) is derived from the analysis of the simplex method. This makes it, to some extent, constructive and allows us to introduce the dual simplex method (Section 1.3). In the second edition the unity of the primal and the dual simplex methods is emphasized. Also new is the way of presentation of the methods for treatment of the methods of solving transportation problems, which are based on the network model, the method of potentials being adjusted to the matrix form.

The discussion of convex programming (Chapter 2) in the second edition is based mainly on the results obtained in linear programming. Furthermore, a finite method for solving convex problems of quadratic programming is given, which is an immediate generalization of the primal simplex method.

In discussing the theory of nonlinear programming (Chapter 3), the same questions as in the first edition are considered, but the proofs are different, drawing on the results of Chapter 1.

We have substantially revised the material on computational methods of nonlinear programming (Chapter 4) and included search methods, in particular. The methods are presented on the basis of the principle of sequential approximation. The basic ideas of other methods of optimization used in solving applied problems are also outlined.

Dynamic programming (Chapter 5) in the second editon is treated as the method of solving special problems, following the general computational methods of nonlinear programming. Using some problems with a clear physical content, we explain the fundamental principles of dynamic programming and various forms of their implementation, accounting for specific characteristics of the problems. The application of dynamic programming to optimal control problems is given in Chapter 7.

The basic results of the classical calculus of variations (Chapter 6) are complemented with some new ones, but, as in the first edition, we consider only conditons for a weak minimum. The conditions for a strong minimum (Chapter 7) are derived from optimal control theory.

The discussion concludes with the treatment of the basic problems of optimal control theory (Chapter 7). In the second edition this material has been revised and extended with new topics, which have arisen in practice only recently.

In our work on the second edition we have been assisted by our collegues at the Optimal Control Methods Department of the Belorussian State University and at the Laboratory of the Theory of Control Processes of the Mathematics Institute of the Academy of Sciences of the Belorussian SSR: O.I. Kostyukova helped in writing Section 2.3 and Sections 5.2 to 5.5; V.M. Raketskij helped with Section 2.4; A.Ya. Kruger wrote Section 3.5; V.V. Gorokhovik wrote Sections 7.3 and 7.8; V.S. Glushenkov derived a new proof of the Bland modification; T.N. Gurina and M.P. Dymkov assisted in preparing the manuscript to publication. To all of them the authors express their gratitude.

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Chapter 1

LINEAR PROGRAMMING

Linear programming refers generally to the class of numerical techniques for maximizing or minimizing linear functions on sets defined by linear equalities and/or inequalities. Linear programming problems were first formulated and studied by the Soviet mathematician L.V. Kantorovich in the 1930s. In the 1940s, the American mathematician G.B. Dantzig developed the simplex method of solution, which opened the way for wide-ranging practical applications, and initiated a whole new area of research as well.

1.1 The Simplex Method

The *simplex method* is the basic computational tool of linear programming.

1.1.1 The canonical problem. Basic solutions

The classical simplex method was developed for the *canonical problem* of linear programming, viz. the maximization problem

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n \rightarrow \max \tag{1}$$

of a linear function in n variables $x_1, x_2, \ldots, x_n$ satisfying m linear equalities

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 , \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 , \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{2}$$

$(b_1, b_2, \ldots, b_m \geq 0)$ and n linear inequalities

$$x_1 \geq 0, \ x_2 \geq 0, \ \ldots , \ x_n \geq 0 . \tag{3}$$

In what follows, we shall normally use matrix notation. Let us introduce the index sets $I = \{1, 2, ..., m\}$ and $J = \{1, 2, ..., n\}$. Then the set of variables $x_1, x_2, ..., x_n$ can be written as the vector $x = x(J) = \{x_j, j \in J\}$. Similarly, $c = c(J) = \{c_j, j \in J\}$ and $b = b(I) = \{b_i, i \in I\}$. The coefficient set $a_{11}, a_{12}, ..., a_{1n}, a_{21}, a_{22}, ..., a_{2n}, ..., a_{m1}, ..., a_{mn}$ is represented conveniently by the matrix $A = A(I, J) = \{a_{ij}, i \in I, j \in J\}$. Operations on vectors and matrices will follow the rules of matrix algebra. Vectors being operated on will be written as column vectors. To represent a row vector, we use the operation of transposition, which we designate by a prime (′). Thus the scalar product of the vectors $c = c(J)$ and $x = x(J)$ is written $c'x$. The expressions $x = 0$ and $x \geq 0$ for a vector x represent the set of componentwise equalities $x_j = 0$, $j \in J$, and inequalities $x_j \geq 0$, $j \in J$.

In our new notation the canonical problem (1)-(3) assumes the compact form

$$c'x \to \max , \qquad Ax = b , \qquad x \geq 0 . \tag{4}$$

It is customary to call the vector c the *cost vector* and its components the *cost coefficients*. Likewise b is called the *constraint vector*, A is the *condition matrix (input matrix)*, and its columns $a_j = A(I, j)$ are the *condition vectors*. The function $c'x$ is called the *objective function* of the problem, the equality

$$Ax = b \tag{5}$$

is the *fundamental constraint*, and the inequality $x \geq 0$ is the *direct constraint* of the canonical problem.

DEFINITION 1. An n-vector x which satisfies every constraint of the problem is called a *feasible solution.*

DEFINITION 2. A feasible solution x^o which solves problem (4),

$$c'x^o = \max c'x , \qquad Ax = b , \qquad x \geq 0 ,$$

is called an *optimal solution.*

The notion of a *basic feasible solution* is essential for the simplex method.

DEFINITION 3. A feasible solution x is called *basic* if $n - m$ of its components are zero, and to the remaining components,

$$x_{j_1}, x_{j_2}, ..., x_{j_m} , \tag{6}$$

there corresponds a linearly independent set of condition vectors

$$a_{j_1}, a_{j_2}, ..., a_{j_m} . \tag{7}$$

We shall call the set $J_B = \{j_1, j_2, ..., j_m\}$ the set of *basic indices*, and $J_N = J \setminus J_B$ the set of *nonbasic indices*. Definition 3 is then equivalent to the following: the feasible solution $x = x(J)$ is basic provided $x_N = x(J_N) = 0$, and $\det A_B \neq 0$, where $A_B = A(I, J_B)$.

The set of vectors (7) is called the *basis* of the basic solution and the matrix A_B consisting of the vectors of the basis, is the *basic matrix*. The components x_j, $j \in J_B$, are called the *basic variables*, and the x_j, $j \in J_N$, are the *nonbasic variables* of the solution x.

REMARK. For a basic solution, the fundamental constraint (5) assumes the form $A_B x_B = b$, where $x_B = x(J_B)$. Therefore the basic solution $x = \{x_B, x_N\}$ can be constructed from the basic matrix A_B: $x_B = A_B^{-1} b$, $x_N = 0$. We can also replace Definition 3 by defining, at the outset, a basic matrix to be any invertible $m \times m$ submatrix A_B of A which satisfies the inequality $A_B^{-1} b \geq 0$, and thus construct a basic solution.

DEFINITION 4. A basic solution is said to be *nondegenerate* if all its basic variables (6) are positive: $x_j \geq 0$, $j \in J_B$.

1.1.2 A Formula for the increment of the objective function

Let x be a basic solution with basic matrix $A_B = A(I, J_B)$. Consider another feasible solution (not necessarily basic) $\bar{x} = x + \Delta x$. We shall find a formula for the increment in the objective function,

$$c'\bar{x} - c'x = c'\Delta x . \qquad (8)$$

By assumption, $Ax = b$, and $A\bar{x} = b$. Hence the change in the solution $\Delta x = \bar{x} - x$ satisfies the equation $A\Delta x = 0$, which componentwise assumes the form

$$A_B \Delta x_B + A_N \Delta x_N = 0 ,$$

$$A_N = A(I, J_N), \quad \Delta x_B = \Delta x(J_B), \quad \Delta x_N = \Delta x(J_N) . \qquad (9)$$

From (9) we obtain

$$\Delta x_B = -A_B^{-1} A_N \Delta x_N \qquad (10)$$

and substitute the result into (8):

$$c'\Delta x = c_B' \Delta x_B + c_N' \Delta x_N = -(c_B' A_B^{-1} A_N - c_N')\Delta x_N , \qquad (11)$$

$$c_B = c(J_B) .$$

We introduce the m-dimensional *potential vector* $u = u(I)$:

$$u' = c_B' A_B^{-1} \tag{12}$$

and the $(n - m)$-dimensional *bound vector* $\Delta_N = \Delta(J_N)$:

$$\Delta_N' = u'A_N - c_N' . \tag{13}$$

Taking (12) and (13) into account, we obtain from (11) for the increment in the objective function:

$$c'\bar{x} - c'x = -\Delta_N' \Delta x_N = -\sum_{j \in J_N} \Delta_j \Delta x_j . \tag{14}$$

From (14) and the general definition of the derivative as a rate of change, we obtain a physical interpretation of the estimate Δ_j: it is the negative rate of change of the objective function at the point x for an increase in the jth nonbasic variable of the basic solution x.

1.1.3 Optimality criterion

Let x be a basic solution with basic matrix A_B. The first question which arises in connection with problem (4) is: When is a given solution optimal? We first compute the bound vector (13) for the solution x.

THEOREM 1 (Optimality criterion). The inequality

$$\Delta(J_N) \geq 0 \tag{15}$$

is a sufficient condition for a basic solution x to be optimal. If x is nondegenerate, the condition is also necessary.

Proof. Sufficiency. By the definition of a basic solution we have $x(J_N) = 0$. From the direct constraint it follows that every solution $\bar{x}$ satisfies the relations

$$\Delta x(J_N) = \bar{x}(J_N) - x(J_N) = \bar{x}(J_N) \geq 0 . \tag{16}$$

Substituting the vectors $\Delta(J_N)$ and $\Delta x(J_N)$ from (15) and (16) into the increment formula (14), we obtain the inequality $c'\bar{x} - c'x \leq 0$, which proves the optimality of the solution x.

Necessity. Suppose that the nondegenerate basic solution x satisfying, by definition,

$$x(J_B) > 0 , \tag{17}$$

does not satisfy inequality (15), i.e., for some $j_0 \in J_N$ the bound Δ_{j_0} is negative:

$$\Delta_{j_0} < 0 . \tag{18}$$

Form the vector $\bar{x} = x + \Delta x$, where Δx is defined as follows. Let the nonbasic components be

$$\Delta x_{j_0} = \Theta \geq 0 , \qquad \Delta x_j = 0 , \qquad j \neq j_0,\ j \in J_N . \tag{19}$$

The basic components are determined from (10):

$$\Delta x(J_B) = -A_B^{-1} A_N \Delta x(J_N) = -\Theta A_B^{-1} a_{j_0} . \tag{20}$$

By (9), for arbitrary Θ the vector $\bar{x}$ satisfies the fundamental constraint: $A\bar{x} = Ax + A\Delta x = Ax = b$. It follows from (19) that the component $\bar{x}(J_N)$ satisfies, for all $\Theta \geq 0$, the direct constraint:

$$\bar{x}(J_N) = x(J_N) + \Delta x(J_N) = \Delta x(J_N) \geq 0 . \tag{21}$$

From (20), for the component $\bar{x}(J_B)$ we get

$$\bar{x}(J_B) = x(J_B) + \Delta x(J_B) = x(J_B) - \Theta A_B^{-1} a_{j_0} . \tag{22}$$

It is seen from (17) that we can find a sufficiently small $\Theta > 0$ such that $\bar{x}(J_B) \geq 0$. Hence, for such a value of Θ the vector $\bar{x}$ is a feasible solution of problem (4). Using (18) and (19) in the increment formula (14) yields the inequality

$$c'\bar{x} - c'x = -\Theta\Delta_{j_0} > 0 , \tag{23}$$

which contradicts the optimality of the solution x. This proves the theorem.

1.1.4 A sufficient condition for the unsolvability of the problem

Suppose that a basic solution x does not satisfy the optimality criterion (15), i.e., for some $j_0 \in J_N$ the bound Δ_{j_0} is negative (cf. (18)). We consider the case where the components x_{jj_0}, $j \in J_B$, of the vector $A_B^{-1} a_{j_0}$ are nonpositive:

$$x_{jj_0} \leq 0 , \qquad j \in J_B . \tag{24}$$

In this case, by (22), the component $\bar{x}(J_B)$ is nonnegative for all $\Theta \geq 0$, i.e., for arbitrary $\Theta \geq 0$ the vector $\bar{x} = \{\bar{x}(J_B), \bar{x}(J_N)\}$ is a feasible solution of problem (4). From (23) it is clear that as Θ increases the value of the objective function for the solution $\bar{x}$ becomes arbitrarily large. We have thus proved the following:

THEOREM 2. Suppose x is a basic solution with a negative bound ($\Delta_{j_0} < 0$) and the corresponding vector $A_B^{-1} a_{j_0}$ has nonpositive components. Then the objective function of problem (4) increases without bound as the variable x_{j_0} of the solution x increases.

1.1.5 Iteration

To continue the analysis of the basic solution x with basic matrix A_B, we consider the case where inequalities (24) fail for every negative bound Δ_{j_0}. It is seen from (23) that if (18) is satisfied, an increase in Θ in (19) yields an increase in the objective function. Hence for problem (4) it is advantageous to choose the largest feasible value of Θ. Componentwise, (22) takes the form $\bar{x}_j = x_j - \Theta x_{jj_0}$, $j \in J_B$; from which it follows that as Θ increases at least one component of the vector $\bar{x}(J_B)$ becomes negative. The components which are zero are those $\bar{x}_j$ which correspond to positive x_{jj_0}, which occurs obviously for

$$\Theta = \Theta_j = \frac{x_j}{x_{jj_0}} . \tag{25}$$

If $\Theta \leq \min \Theta_j$ and $x_{jj_0} > 0$, then all of the numbers in (25) are nonnegative, and the vector $\bar{x} = \{\bar{x}(J_B), \bar{x}(J_N)\}$ is a feasible solution of (4). If $\Theta > \min \Theta_j$ and $x_{jj_0} > 0$, then the vector $\bar{x}$ has negative components. Hence the largest possible Θ° for which the vector $\bar{x} = x + \Delta x$ is a feasible solution is

$$\Theta^\circ = \Theta_{i_0} = \frac{x_{i_0}}{x_{i_0 j_0}} = \min_{\substack{x_{jj_0}>0 \\ j \in J_B}} \frac{x_j}{x_{jj_0}} . \tag{26}$$

If the basic solution x is nondegenerate (i.e., $x_j > 0$, $j \in J_B$), then

$$\Theta^\circ > 0 . \tag{27}$$

Suppose we exchange the basic solution x for another feasible solution $\bar{x} = x + \Delta x$, where Δx is the vector with the components defined in (19) and (20), with $\Theta = \Theta^\circ$. Then by (23) the objective function has increased by the amount $-\Delta_{j_0}\Theta^\circ \geq 0$, which, by virtue of (18), (27) is positive if the original solution x was nondegenerate.

We shall show that $\bar{x}$ is a basic solution. Among the components $\bar{x}_j$, $j \in J_N$, only one can be positive: $\bar{x}_{j_0} = \Theta^\circ$. On the other hand, among the components $\bar{x}_j$, $j \in J_B$, one component $\bar{x}_{i_0}$ must be zero, since by (25) and (26)

$$\bar{x}_{i_0} = x_{i_0} - \Theta^\circ x_{i_0 j_0} = x_{i_0} - \frac{x_{i_0} x_{i_0 j_0}}{x_{i_0 j_0}} = 0 .$$

Thus the $n - m$ variables

$$\bar{x}_j\,, \qquad j \in \bar{J}_N\,, \quad \bar{J}_N = (J_N \setminus j_0) \cup i_0\,,$$

of the solution $\bar{x}$ are zero. To the remaining variables $\bar{x}_j$, $j \in \bar{J}_B$, where $\bar{J}_B = (J_B \setminus i_0) \cup j_0$, there correspond the condition vectors

$$a_j\,, \qquad j \in \bar{J}_B\,. \tag{28}$$

Let u_{ij}, $i \in J_B$, $j \in I$, be the entries of the matrix A_B^{-1}. By definition, the jth column of the matrix consists of the coefficients of the representation of the unit vector $e_j = \{0, 0, \ldots, 0, \underset{j}{1}, 0, \ldots, 0\}$ as a linear combination of the columns of the matrix $A_B = \{a_i, i \in J_B\}$:

$$\sum_{i \in J_B} a_i u_{ij} = e_j\,, \qquad j \in I\,. \tag{29}$$

We write the vector a_{j_0} as a linear combination of the basis (7):

$$\sum_{i \in J_B} a_i x_{ij_0} = a_{j_0}\,. \tag{30}$$

By (26), the number $x_{i_0 j_0}$ is positive. Hence from (30) we can find

$$a_{i_0} = \frac{1}{x_{i_0 j_0}} a_{j_0} - \sum_{j \in J_B \setminus i_0} a_i \frac{x_{ij_0}}{x_{i_0 j_0}}\,. \tag{31}$$

Substitute (31) into (29):

$$\frac{u_{i_0 j}}{x_{i_0 j_0}} a_{j_0} + \sum_{j \in J_B \setminus i_0} a_i \left(u_{ij} - \frac{u_{i_0 j}\, x_{ij_0}}{x_{i_0 j_0}} \right) = e_j\,, \qquad j \in I\,.$$

These equalities show that the vectors (28) are linearly independent, and the entries $\bar{u}_{ij}$, $i \in \bar{J}_B$, $j \in I$, of the matrix $\bar{A}_B^{-1}$, the inverse of the new basic matrix $\bar{A}_B = A(I, \bar{J}_B)$, are given by

$$\begin{aligned} \bar{u}_{j_0 j} &= \frac{u_{i_0 j}}{x_{i_0 j_0}}\,, && j \in I\,, \\ \bar{u}_{ij} &= u_{ij} - \frac{x_{ij_0}\, u_{i_0 j}}{x_{i_0 j_0}}\,, && i \neq j_0,\ i \in \bar{J}_B,\ j \in I\,. \end{aligned} \tag{32}$$

Therefore $\bar{x}$ is a basic solution with basic matrix $\bar{A}_B = A(I, \bar{J}_B)$, $\bar{J}_B = (J_B \setminus i_0) \cup j_0$.

The transformation $x \to \bar{x}$ from the old basic solution x to the new basic solution $\bar{x}$ is called *simplex iteration*. The calculations above show that the iteration can be done on a computer in a finite number of steps.

REMARK. The iteration above began with an arbitrary negative bound Δ_{j_0}. If the number of nonbasic variables $|J_N|$ is small, the following choice is often preferable:

$$\Delta_{j_0} = \min \Delta_j , \qquad j \in J_N . \tag{33}$$

If the number of nonbasic variables is large, (33) can prove to be tedious. Fortunately, there are also other methods for choosing Δ_{j_0}

The sequential transformation of the basic solution by means of simplex iterations constitutes the simplex method of linear programming.

1.1.6 The simplex algorithm. The multiplicative method

The matrix A_B^{-1}, the inverse of the basic matrix, played a major role in the calculations of Sections 1.1.2–1.1.5. We shall therefore describe an algorithm for solving problem (4) in terms of A_B^{-1}.

At the first stage, the initial inverse matrix is A_B^{-1}.

At the kth stage, suppose the matrix is $(A_B^{-1})_k = ([A(I, J_B^k)]^{-1})_k$. The basic solution corresponding to the matrix $(A_B^{-1})_k$ is $x = \{x(J_B^k) = (A_B^{-1})_k b,\ x(J_N^k) = 0\}$, $J_N^k = J \setminus J_B^k$.

1) We find the vector $u'(I) = c'(J_B^k)(A_B^{-1})_k$.

2) We find the bounds $\Delta_j = u'(I)A(I, j) - c_j$, $j \in J_N^k$.

3) If there are no negative bounds among all the Δ_j, $j \in J_N^k$, then the process terminates at an optimal solution x.

4) If the set of Δ_j, $j \in J_N^k$, contains negative bounds, then from among them we choose $\Delta_{j_0} < 0$, $j_0 = j_0(k)$.

5) We compute the components x_{jj_0} of the $(A_B^{-1})_k A(I, j_0)$, $j \in J_B^k$.

6) If there are no positive components x_{jj_0}, $j \in J_B^k$, then the process stops: the objective function grows without bound as the j_0th component of x increases.

7) For each positive x_{jj_0}, $j \in J_B^k$, we calculate the numbers $\Theta_j = x_j / x_{jj_0}$ and pick the smallest among them, $\Theta^\circ = \Theta_{i_0}$, $i_0 = i_0(k)$.

8) We replace J_B^k, J_N^k by the new sets $J_B^{k+1} = (J_B^k \setminus i_0) \cup j_0$; $J_N^{k+1} = (J_N^k \setminus j_0) \cup i_0$, and using (32) we compute the elements $\bar{u}_{ij}$, $i \in \bar{J}_B$, $j \in I$, of the matrix $(A_B^{-1})_{k+1}$, assuming that in (32) $J_B = J_B^k$; u_{ij}, $i \in J_B^k$, $j \in I$, are elements of $(A_B^{-1})_k$. This completes the kth step of the iteration.

A flow chart of the algorithm is shown in Figure 1.1.

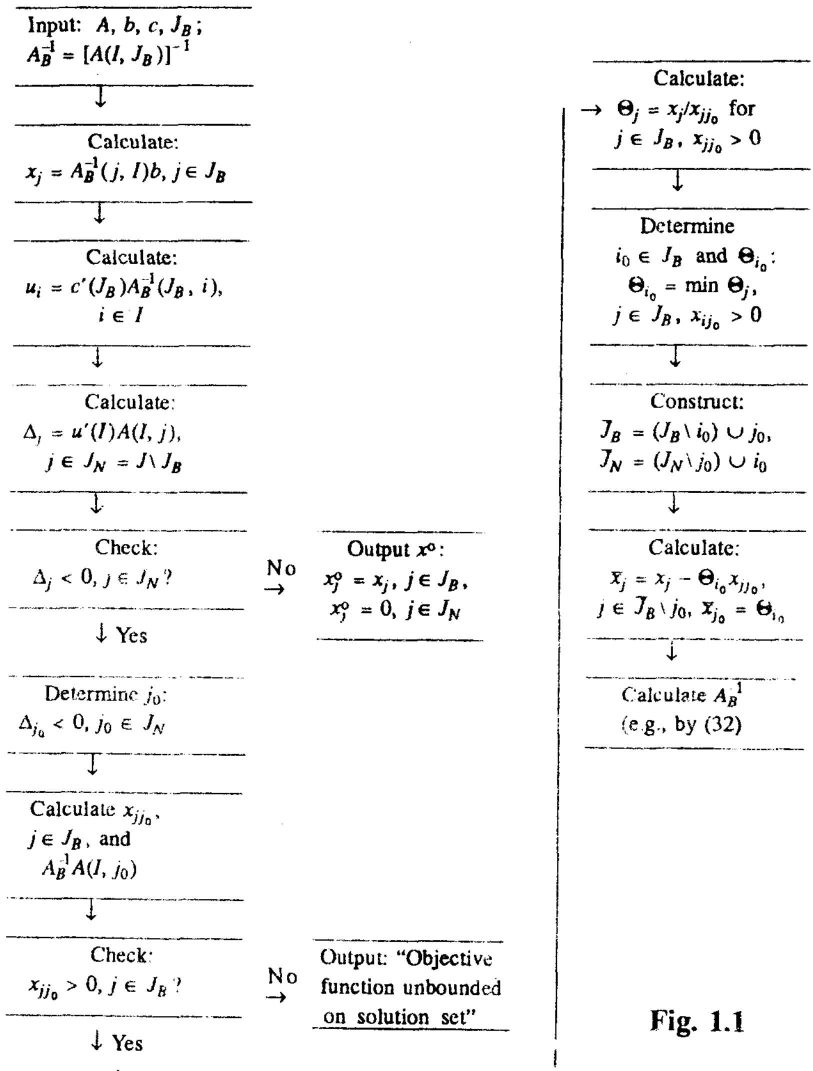

Fig. 1.1

At each step of the algorithm we update the $m \times m$ matrix A_B^{-1}. The matrix inversion can be avoided by a widely used variation of the simplex method called the *multiplicative method*. It is based on the fact that the matrices $(A_B^{-1})_{k+1}$ and $(A_B^{-1})_k$ are related, by virtue of (32), by

$$(A_B^{-1})_{k+1} = D_k (A_B^{-1})_k , \tag{34}$$